MULTIPLE CRITERIA DECISION MAKING
Vol. 17 2022

Somdeb Lahiri*

AXIOMATIC CHARACTERIZATIONS OF PROBABILISTIC MAX-MIN EXTENDED CHOICE CORRESPONDENCE

DOI: 10.22367/mcdm.2022.17.05

Received: 24.11.2022 | Revised: 5.01.2023 | Accepted: 1.03.2023.

Abstract

In this paper we provide two axiomatic characterizations of the probabilistic max-min extended choice correspondence support, for a decision maker who has state-dependent preferences (represented by a linear order) over the set of alternatives and a (subjective) probability vector over states of nature, where both preferences and probability vectors are variable.

Keywords: state-dependent preferences, extended choice correspondence.

1 Introduction

A (fixed agenda) extended choice correspondence assigns to each profile of state-dependent strict rankings over the set of alternatives and probability vector over a non-empty finite set of states of nature, a non-empty (not necessarily proper) subset of alternatives, from a given non-empty finite and fixed set of alternatives. The genesis of this concept and a fairly detailed mathematical discussion of it can be found in Lahiri (2020/2021). With a different interpretation, Denicolo (1985) refers to a special case of the same mathematical entity as a social choice correspondence. The special case corresponds to equiprobable states of nature, but since the interpretation in the paper by Denicolo cited above is one of group decision-making under certainty, there is a point at which the analysis in our paper would remain incomplete, had we interpreted the framework differently. The different states of nature could be interpreted as different criteria and the probability vectors as weights, thus reducing it to a multi-criteria

* (Former Professor) PDE University, India and (Adjunct Professor) LJ University, India, e-mail: somdeb.lahiri@gmail.com, ORCID: 0000-0002-5247-3497.

decision making (MCDM) problem with a weight for each criterion ("MCDM with weights"). However, we don't want to push that interpretation further and would rather root for interpreting "MCDM with weights" as decision-making under probabilistic uncertainty.

The framework introduced in Lahiri (2020/2021) is an extension of the seminal model of social choice theory developed by Kenneth J. Arrow. A decision maker is faced with making a choice under probabilistic uncertainty (risk) in which uncertainty is with regard to a future state of nature, which is realized after the decision has been made. The decision maker is provided with (or aware of) a data profile, which is a pair whose first component is a profile of state-dependent rankings over (the consequences) a non-empty finite set of alternatives and whose second component is a probability distribution over a non-empty finite set of states of nature. A decision support system (DSS) or decision aid is required to choose a non-empty "desirable" set of alternatives from which the final choice has to be made. The decision aid or DSS has no bias in favor of any one or more alternatives that it suggests. Such a decision support system is called an extended choice correspondence, i.e., a rule which associates with each data profile from a given set of data profiles a non-empty finite set of desirable alternatives.

The problem of choosing one or more alternatives from a given set of alternatives was raised and rigorously formulated for the first time in a seminal contribution on majority voting by Pattanaik (1970). For the classical theory of decision making under uncertainty in the state dependent case − which is the other and major motivation behind Lahiri (2020/2021) − one may refer to Karni (1985).

The initial concern that led to the frameworks discussed in this paper is that Arrowian voting theory framework does not have anything to say about the role of negotiations in group decision making and may therefore be very inadequate for our understanding of decision making in society. In view of this, slight extensions of voting models as models of choice under risk may serve a useful purpose.

The reasons for our interest in state-dependent preferences are precisely the same as the ones discussed in Karni (1985), i.e., it is so obviously true that it does not need justification beyond citing trivial day-to-day examples as Karni has done in his book. However, Karni focuses on state-dependent utility functions and it is our contention here that the informational requirements for (state-dependent?) utility of state-dependent monetary surplus derived by decision makers from consuming alternatives, may prove prohibitive and a significant reason for "bounded rationality", thus leading to "useable" preferences being represented by rankings instead of utility functions. Knowledge of the exact state-dependent monetary surplus (and not necessarily the state-dependent utility functions) is not easy to obtain for the purpose of decision making, not only because the cost of obtaining such information is often exorbitant, but also because the knowledge of the monetary benefit from the chosen alternative may only be available on conditions prevailing at a future date that are neither accessible nor can be experienced at the time the decision has to be made.

Hence, the major justification for the framework and the investigation in Lahiri (2020/2021) is that the classical theory of decision making under uncertainty that rests on the assumption of maximization of expected utility (state-dependent or not) has an important limitation − i.e. the decision maker's preferences may not be "useable" in the form of cardinal utility functions, but only as rankings. That leads to a departure from the classical theory and opens up the possibility of decision makers using other algorithms (decision aids) for the purpose of decision making under risk. That is the line of investigation pursued in this paper. A full-fledged application using components of this framework to prove the existence of "preferred with probability at least half winners" has been provided in Lahiri (2020; 2021). This however, is not meant to be a denial of the worth of the huge literature based on utility functions that uses procedures other than expected utility maximization, to explain paradoxes that arise if the latter criterion is used to explain decision making under uncertainty. One such is the work of Gilboa (1988) which suggests that decision makers maximize a function that is increasing in both expected utility of an alternative and the worst utility of the alternative, in arriving at optimal choices. Such procedures would require information about the state-dependent utility of each alternative, and it is our contention here − as observed earlier − that such information may not be as easily available as expected utility theory presupposes. The ordinal equivalent of the procedure suggested by Gilboa (1988) would require maximizing a function of the Probabilistic Borda Score of an alternative (see Lahiri (2020/2021) and the worst rank that the alternative attains with positive probability, where the function is "increasing" in the first variable and "decreasing" in the second.

Here we begin by setting up the model for extended choice correspondences. In this framework we provide two axiomatic characterizations of the probabilistic max-min extended choice correspondence. This extended choice correspondence is based on the max-min choice correspondence due to Campbell, Kelly and Qi (2018). The max-min choice correspondence of Campbell, Kelly and Qi (2018) selects for each preference profile those alternatives which have the best "worst rank". In our framework, for a data profile − a pair comprising a strict preference profile and a probability vector (for the states of nature) − a "max-min alternative" is an alternative whose worst rank among states of nature that occur with probability is the best. The worst rank of a max-min alternative is said to be the "max-min rank". Our probabilistic max-min extended choice correspondence selects for each data profile those max-min alternatives which have the least positive probability of attaining the "max-min rank". We ignore those states of nature that occur with probability zero, since if an alternative attains its worst rank with probability zero, it is improbable (though not impossible) that it will attain such a rank. Furthermore, if a max-min alternative attains the max--min rank with lowest probability, then it attains a superior rank with the highest probability among all max-min alternatives. It is very unlikely that a risk-averse individual, to whom the probabilistic max-min extended choice correspondence would be recommended, could wish for anything better. A related earlier paper is the one by Congar and Merlin (2012), where the main concern is with axiomatic characterization of max-min "social welfare function". The domain of the probabilistic max-min extended choice correspondence, whose axiomatization we provide, is the set of all data profiles, such that for any non-empty subset of probability vectors, all strict preference profiles can be associated with any probability vector in the subset. The strict sub-domain where the data profiles are such that those states of nature that occur with positive probability have equal probability of occurrence is said to be one with equiprobable support. On the domain with equiprobable support, our solution concept is a refinement of the one discussed in Campbell, Kelly and Qi (2018), with a different interpretation. This would correspond to the sub-solution of the one in Campbell, Kelly and Qi (2018), where only those max-min alternatives with max-min ranks for the fewest number of agents are chosen.

Our study here concerns decision making under uncertainty and one of the earliest works dealing with axiomatic characterizations in such a scenario is that of Maskin (1979). However, the structure of the underlying set of alternatives from which choices are made in Maskin (1979) is completely different from what we assume here and hence our axiomatization, as well as the methodology we use to obtain our results, is completely different from the corresponding ones that are reported and used here. Another notable contribution in a related but different line of research is the work of Gilboa and Schmeidler (1989). A fairly comprehensive survey of research on decision-making uncertainty is the paper by Kelsey and Quiggin (1992). A paper that could have been an exact predecessor to our work here is the one due to Puppe and Schlag (2009), if they had used state dependent strict rankings (even rankings would do!) instead of state dependent pay-off functions. The fact that in their context, the set of alternatives from which choices in a state of nature can be made is state-dependent *may not* be a problem, if we take the given fixed set of alternatives to be the union of sets of alternatives available over the different states of nature and in each state of nature ranked those alternatives that are not available in that state of nature, strictly below those that are available in that state of nature. The paper by Congar and Merlin (2012) which is concerned with the max-min (Rawlsian) social welfare function, is related to the work that follows, but may fail to shed light on our results, since it uses a variable number of states (voters) argument via two axioms − duplication and weak separability − in the analysis reported there.

In a final section of this paper we provide an example to illustrate how "bounded rationality" arises in the context of probabilistic uncertainty due to absence of sufficient information about the state-dependent monetary surpluses of an alternative, thereby rendering expected utility maximization practically unusable. Under such circumstances, using a framework of analysis based on state-dependent rankings of alternatives may be unavoidable if not inevitable.

Proofs of results are available from the author on request.

2 The framework of analysis

The following framework is a fairly close adaptation of the ones available in Denicolo (1985) and section 2.2 of Endriss (2011) and discussed thoroughly in Lahiri (2020/2021).

Consider a decision maker (DM) faced with the problem of choosing one or more alternatives from a non-empty finite set of alternatives X, containing at least three elements. Let $\Psi(X)$ denote the set of all non-empty subsets of X. For a positive integer $n \geq 3$, let $N = \{1, 2, \ldots, n\}$. Contrary to convention we will refer to an element in N as a state of nature and to the set N as the set of states of nature.

A **strict preference relation/strict ranking** on X is a linear order (i.e. a reflexive, complete/connected/total, transitive and anti-symmetric binary relation) on X. Generally, a strict preference relation is denoted by R with P denoting its asymmetric part. If for $x, y \in X$, it is the case that $(x, y) \in R$, then we shall denote it by xRy and say that x is **at least as good as** y **for the strict preference relation** R. Similarly, xPy is interpreted as x is **strictly preferred** to y **for the strict preference relation** R.

Given a strict preference R and an alternative x, **the rank of** x **at** R, denoted by $rk(x, R) = |\{y \in X|yRx\}|$, i.e., 1 + cardinality of the set of alternatives strictly preferred to x for the strict preference relation R.

Let $\mathcal{L}$ denote the set of all strict preference relations on X.

A **strict preference profile**, denoted by R_N, is a function from N to $\mathcal{L}$. R_N is represented as the array $\langle R_i|i \in N\rangle$, where R_i is the strict preference relation/strict ranking in state of nature i. The set of all preference profiles is denoted by $\mathcal{L}^N$.

A **probability vector over** N is a vector $p \in \mathbb{R}_+^N$ satisfying $\sum_{i=1}^{N} p_i = 1$ where for $i \in N$, p_i is the probability that state of nature 'i' occurs.

The **set of probability vectors over** N is denoted by Δ.

Given a probability vector p, the set $\{j|p_j > 0\}$ is said to be **the support of** p and is denoted by support(p).

Since probabilities are associated with events, for each $i \in N$, the state of nature i represents a non-empty set and N is a finite partition of some underlying sample space.

Given $(R_N, p) \in \mathcal{L}^N \times \Delta$ and an alternative x (i.e. $x \in X$), a state of nature i (i.e., $i \in N$) will be said to be **a worst state of nature for** x at (R_N, p) if $i \in \text{argmax}_{j \in \text{support}(p)}\, rk(x, R_j)$.

The above definition says that a state of nature is a worst state of nature for an alternative if the state of nature occurs with "positive probability" and the alternative does not attain any worse rank with "positive probability".

Given $(R_N, p) \in \mathcal{L}^N \times \Delta$ and an alternative x (i.e. $x \in X$), the set $WS(x, (R_N, p)) = \{i|$ i is a worst state of nature for $x\}$ is said to be **the set of worst states of nature** for x at (R_N, p), and for $i \in WS(x, (R_N, p))$, $rk(x, R_i)$, denoted worstrk$(x,(R_N, p))$, is said to be **the worst rank** of x at (R_N, p).

Clearly, worstrk$(x,(R_N, p)) = \max\{rk(x,R_i)|i \in \text{support}(p)\}$ for all $x \in X$.

For all $(R_N, p) \in \mathcal{L}^N \times \Delta$, let $Mm(R_N, p) = \text{argmin}_{y \in X}\, \text{worstrk}(y, (R_N, p))$.

$Mm(R_N, p)$ is said to be **the set of max-min alternatives at** (R_N, p). The **max-min rank for** (R_N, p) is equal to the unique worstrk$(x,(R_N, p))$ for any $x \in Mm(R_N, p)$.

A **domain** is any non-empty subset of $\mathcal{L}^N \times \Delta$. We will denote a domain by $\mathcal{R}$.

An **extended choice correspondence** (ECC) on (domain) $\mathcal{R}$ is a function f from $\mathcal{R}$ to Ψ(X).

The problem with $Mm(R_N, p)$ and any ECC that does not discriminate between states of nature which have positive probability is that they might overemphasize the "extremely unlikely" to absurd extents thereby denying the decision maker the right to exercise one's discretion within reasonable limits.

Example 1: X = {x,y}, n = 2, $p_1 = \frac{1}{100}$, $p_2 = \frac{99}{100}$.

$rk(x,R_1) = 1$, $rk(y,R_1) = 2$; $rk(x,R_2) = 2$, $rk(y,R_2) = 1$.

$Mm(R_N, p) = \{x,y\}$. But, does 'x' have any reason to be treated at par with 'y', when there is a 99% chance that 'y' is going to be preferred to 'x'?

Hence, we consider the following procedure.

The following notation will prove useful in what follows.

Given $(R_N, p) \in \mathcal{L}^N \times \Delta$ and $x \in X$, the **probability of the worst rank** of x at (R_N, p) denoted by $Pr(WS(x, R_N, p)) = \sum_{i \in WS(x,R_N,p)} p_i$.

An ECC on $\mathcal{R}$ is said to be the **probabilistic max-min choice correspondence**, denoted by f^{PMm}, if for all $(R_N, p) \in \mathcal{R}$, $f^{PMm}(R_N, p) = \{x \in Mm(R_N, p) | Pr(WS(x, R_N, p)) \leq Pr(WS(y, R_N, p))$ for all $y \in Mm(R_N, p)\}$, i.e., $f^{PMm}(R_N, p)$ is the set of max-min alternatives with least total probability of securing the best worst rank at (R_N, p).

Thus, an ECC is f^{PMm} which at any (R_N, p) in the domain of the ECC, chooses those max-min alternatives whose max-min rank occur with least probability, i.e., the chosen alternatives are those max-min alternatives that each occurs at its worst rank with the least probability. In other words, f^{PMm} minimizes "the probability" with which a max-min rank occurs.

Clearly, $f^{PMm}(R_N, p)$ for Example 1 is {y}.

Choosing a "best ranked" alternative from among those which attain its worst rank with least probability may prove to be misleading as the following example reveals.

Example 2: X = {x,y,z}, n = 3, $p_1 = \frac{1}{100}$, $p_2 = \frac{98}{100}$, $p_3 = \frac{1}{100}$.

$rk(x,R_1) = 1$, $rk(y,R_1) = 2$, $rk(z,R_1) = 3$; $rk(x,R_2) = 2$, $rk(y,R_2) = 1$, $rk(z,R_2) = 3$; $rk(x,R_3) = 3$, $rk(y,R_3) = 2$, $rk(z,R_3) = 1$.

The probability with which x gets its worst rank, i.e. 3, is $\frac{1}{100}$. The probability with which y gets its worst rank, i.e. 2, is $\frac{2}{100} = \frac{1}{50}$. The probability with which z gets its worst rank, i.e. 3, is $\frac{99}{100}$. Hence (in this case) the unique alternative which attains its worst rank with least probability is x and the worst rank is equal to 3.

However, there is only one max-min ranker, i.e. y, and the first method selects 'y'. This seems quite reasonable for a risk-averse individual, since there is a 99% chance that 'y' will be preferred to 'x' and a 99% chance that 'y' will be preferred to 'z'.

In view of the fact that the domain $\mathcal{R}$ is a subset of $\mathcal{L}^N \times \Delta$, given any $(R_N, p) \in \mathcal{R}$ it is not possible for two different alternatives to have the same worst state of nature at (R_N, p).

In what follows we will be concerned only with those domains which satisfy the following property:

Domain Property: $\mathcal{R} = \mathcal{L}^N \times Q$, where Q is a non-empty subset of Δ.

3 Some axioms and a lemma that will be useful on the way

We begin this section with two very desirable axioms that few would wish to contest.

An ECC f on $\mathcal{R}$ is said to satisfy **Unanimity** if for $(R_N, p) \in \mathcal{R}$, $x \in X$: $[rk(x, R_i) = 1$ for all $i \in N]$ implies $[f(R_N, p) = \{x\}]$.

An ECC f on $\mathcal{R}$ is said to satisfy **Independence of Irrelevant States** (to be **Independent of Irrelevant States**) (**IIS**) if for all $(R_N, p), (R'_N, p) \in \mathcal{R}$: $[\{j|p_j > 0\} \subset \subset \{j|R_j = R'_j\}]$ implies $[f(R'_N, p) = f(R_N, p)]$.

The next axiom is considerably more specific to our present context.

An ECC f on $\mathcal{R}$ is said to satisfy the **Worst-Rank Property** if for all $(R_N,p) \in \in \mathcal{R}$ and $x \in f(R_N,p)$: [worstrk(x, (R_N,p)) > 1] implies [for no $y \in X$ is it the case that worstrk(y, (R_N,p)) = worstrk(x, (R_N,p)) − 1].

An ECC f on $\mathcal{R}$ is said to satisfy **Worst-Rank Positive Responsiveness (W-RPR)** if for all $(R_N,p), (R'_N,p) \in \mathcal{R}$, $x \in f(R_N,p)$ satisfies worstrk(x, (R_N,p)) > 1 and $i \in$ WS(x,(R_N,p)): [(a) $R'_k = R_k$ for all $k \neq i$; (b) rk(x,R'_i) = rk(x, R_i) − 1, rk(z, R'_i) = rk(z, R_i) if $z \neq x$ and rk(z, R_i) ≠ rk(x, R_i) – 1] implies [f(R'_N,p) = {x}].

From the construction of (R'_N,p) it is clear that rk(z, R_i) = rk(x, R_i) – 1 implies rk(z, R'_i) = rk(z, R_i) + 1 = rk(x, R_i).

Note that from the definition of W-RPR, we get either WS(x,(R'_N,p)) = = WS(x,(R_N,p))\{i} in which case worstrk (x,(R'_N,p)) = worstrk(x,(R_N,p)) and $\sum_{j \in WS(x,(R'_N,p))} p_j = \sum_{j \in WS(x,(R_N,p))} p_j - p_i < \sum_{j \in WS(x,(R_N,p))} p_j$ or WS(x,(R'_N,p)) = = WS(x,(R_N,p)) in which case worstrk (x,(R'_N,p)) = worstrk (x,(R_N,p)) – 1.

W-RPR says that if i is a worst state of nature for some chosen alternative with worst rank greater than 1 and if in state of nature i this alternative exchanges its position with the alternative immediately above it at ‘i’, then after such a change this alternative becomes the uniquely chosen alternative and the unique max-min alternative.

As an immediate consequence of Unanimity, IIS, Worst Rank Property and W-RPR is the fact that chosen alternatives must be max-min alternatives.

Lemma 1: If an ECC f on a domain $\mathcal{R}$ satisfies Unanimity, IIS and W-RPR then for all $(R_N,p) \in \mathcal{R}$, it must be the case that $f(R_N, p) \subset Mm(R_N, p)$.

4 The main result

An ECC f on $\mathcal{R}$ is said to satisfy **Not Chosen After Worser Rank (NCWR)** if for $(R_N, p), (R'_N, p) \in \mathcal{R}$, $x \notin f(R_N, p)$ and $i \in N$: [(a) $R'_k = R_k$ for all $k \neq i$; (b) rk(x, R'_i) = rk(x, R_i) + 1, rk(z, R'_i) = rk(z, R_i) if $z \neq x$ and rk(z, R_i) ≠ rk(x, R_i) + 1] implies [$x \notin (R'_N, p)$].

NCWR says that if initially an alternative is not chosen, then it remains unchosen if in any state of nature, it exchanges places with an alternative ranked immediately below it. It is easily verified that f^{PMm} satisfies NCWR.

An ECC f on $\mathcal{R}$ is said to satisfy **Not More Probable Worse Rank (NMPWR)** if for all (R_N,p), $(R'_N, p) \in \mathcal{R}$, $x \in X$ and $i \in I$: $[x \in f(R_N, e^I) \cap f(R'_N, e^I)]$ implies $[P(WS(x, (R'_N, p))) \leq P(WS(y, (R'_N, p)))$ for all $y \in f(R'_N, p)]$, where: [(a) $R'_k = R_k$ for all $k \neq i$; (b) $rk(x,R'_i) = rk(x, R_i) + 1$, $rk(z, R'_i) = rk(z, R_i)$ if $z \neq x$ and $rk(z, R_i) \neq rk(x, R_i) + 1$].

From the construction of (R'_N,p) it is clear that $rk(z, R_i) = rk(x, R_i) + 1$ implies $rk(z, R'_i) = rk(z, R_i) – 1 = rk(x, R_i)$.

NMPWR says that if a chosen alternative is chosen after it exchanges its position with the alternative immediately below it at a state of nature occurring with positive probability, then at the lower rank it occurs with positive probability not more often than any other chosen alternative does.

An ECC f on $\mathcal{R}$ is said to satisfy **Greater Probability if Exclusion After Worsening (GPEAW)** if for all (R_N,p), $(R'_N,p) \in \mathcal{R}$, $x \in X$ and $i \in N$: $[\{x\} = Mm(R_N,p)$ and $x \notin f(R'_N,p)]$ implies $[P(WS(x, (R'_N,p))) > P(WS(y, (R'_N,p)))$ for some $y \in f(R'_N,p)]$, where: [(a) $R'_k = R_k$ for all $k \neq i$; (b) $rk(x,R'_i) = rk(x, R_i) + 1$, $rk(z, R'_i) = rk(z, R_i)$ if $z \neq x$ and $rk(z, R_i) \neq rk(x, R_i) + 1$].

From the construction of (R'_N,p) it is clear that $rk(z, R_i) = rk(x, R_i) + 1$ implies $rk(z, R'_i) = rk(z, R_i) – 1 = rk(x, R_i)$.

Given Lemma 1, GPEAW is the converse of NMPWR. Along with Lemma 1, what NMPWR and GPEAW together say is the following:

A chosen alternative is chosen after it exchanges its position with the alternative immediately below it at a state of nature occurring with positive probability if and only if at the lower rank it occurs with positive probability not more often than any other chosen alternative does.

Note: By IIS, the three properties NCWR, NMPWR and GPEAW hold non-vacuously only when the state of nature ‘i’ in their definitions belong to support(p).

Proposition 1: If an ECC f on $\mathcal{R}$ satisfies Unanimity, IIS, Worst Rank Property, W-RPR, NCWR NMPWR and GPEAW then for all $(R_N,p) \in \mathcal{R}$, $f(R_N,p) = f^{PMm}(R_N,p)$.

It is easy to see that on the domain with equiprobable support f^{PMm} satisfies Unanimity, IIS, Worst Rank Property, NCWR, W-RPR, NMPWR and GPEAW.

Thus we arrive at the following theorem.

Theorem 1: An ECC f on $\mathcal{R}$ satisfies Unanimity, IIS, Worst Rank Property, W-RPR, NCWR, NMPWR and GPEAW if and only if $f = f^{PMm}$ on $\mathcal{R}$.

An alternative axiomatic characterization with a shorter proof can be obtained by replacing NMPWR by the following property.

An ECC f on $\mathcal{R}$ is said to satisfy **Non-Domination of Worst Rank (ND-WR)** if there does not exist $(R_N, p) \in \mathcal{R}$ and $x,y \in f(R_N, p)$ satisfying worstrk$(x, R_N, p) \leq$ worstrk(y, R_N, p), Pr.(WS$(x, R_N, p)) \leq$ Pr.(WS(y, R_N, p)) with at least one strict inequality.

ND-WR says that given two chosen alternatives if one has a "better" worst rank than the other, then the probability of the first alternative securing its worst probable rank must be greater than the corresponding probability of the second.

With ND-WR replacing NMPWR, we have the following.

Proposition 2: If an ECC f on $\mathcal{R}$ satisfies Unanimity, IIS, Worst Rank Property, W-RPR, NCWR ND-WR and GPEAW then for all $(R_N,p) \in \mathcal{R}$, $f(R_N,p) = f^{PMm}(R_N,p)$.

Since f^{PMm} satisfies ND-WR, we get the following result.

Theorem 2: An ECC f on $\mathcal{R}$ satisfies Unanimity, IIS, W-RPR, NCWR, ND-WR and LPEAW if and only if $f = f^{PMm}$ on $\mathcal{R}$.

5 Bounded rationality in decision aiding − an example

This section was included at the behest of Professor Tadeusz Trzaskalik and I thank him for the suggestion.

Consider an individual who has to book a room in a hotel for an overnight stay that is supposed to take place a week later. There are three types of rooms in the hotel: Rooms with air conditioners "x", Rooms with air coolers "y" and Rooms with just a ceiling fan "z". The tariff for a room of type x is INR 3500 per night, for a room of type y it is INR 3000 per night, and for a room of type z it is INR 2500. The weather during the night of the proposed stay at the hotel could be either "1" hot and dry, "2" hot and humid, or "3" just pleasant, with equal probability of occurrence of each of the three types of weather.

The individual's satisfaction from each of the three types of rooms is reflected in a reservation price which depends not only on the weather but on other amenities (such as the availability of air freshener, room service etc.) and *in particular*, "the intensity of the weather condition", about which information is not available to the individual at the time of booking the room. This is a situation that may be referred to as "bounded rationality due to lack of sufficient information". However, on the basis of the information available − which includes room tariffs − the individual's weather-dependent preferences are as follows:

If the weather is as in 1, y is ranked first, x is ranked second and z is ranked third.

If the weather is as in 2, x is ranked first, z is ranked second and y is ranked third.

If the weather is as in 3, z is ranked first, y is ranked second and x is ranked third.

On the basis of the above information the individual prefers x to z with probability $\frac{2}{3}$, y to x with probability $\frac{2}{3}$ and z to y with probability $\frac{2}{3}$. This situation is referred to as Condorcet Paradox, where the individual is indecisive due to lack of sufficient information. Almost all reasonable ECC would recommend {x,y,z} under such circumstances.

However, if the "reservation price" for a type of room w ∈ {x,y,z} under the weather condition j ∈ {1,2,3} is denoted by WTP(w, j) (i.e. willingness to pay for w if the weather is as in "j") then the individual's expected surpluses for a type of room x is given by $\frac{1}{3}$[WTP(x, 1) + WTP(x, 2) + WTP(x, 3)] − 3500, for a type of room y is given by $\frac{1}{3}$[WTP(y, 1) + WTP(y, 2) + WTP(y, 3)] − 3000 and for a type of room z is given by $\frac{1}{3}$[WTP(z, 1) + WTP(z, 2) + WTP(z, 3)] − 2500.

It is not unreasonable to assume that WTP(x, 1) = 4000, WTP(y, 1) = 3600, WTP(z, 1) = −1000; WTP(x, 2) = 4500, WTP(y, 2) = 3000, WTP(z, 2) = 3000; and WTP(x, 3) = 3600, WTP(y, 3) = 3500, WTP(z, 3) = 3500. However, such information will be available only after arrival at the hotel and not at the time of booking a room.

During a night that is not humid, an air cooler can be made to serve exactly the same purpose as a ceiling fan, simply by turning off the water pump of the air cooler. On such nights the surplus (i.e., reservation price minus room tariff) is clearly greater for a room with a ceiling fan than for a room with an air cooler. It is only on a hot and dry night that the ceiling fan has the same effect as a "blast furnace".

Thus, on a hot and dry night the individual's surplus from a room of type x is 500, from a room of type y is 600 and from a room of type z is −3500.

On a hot and humid night, the individual's surplus from a room of type x is 1000, from a room type of room y is 0 and from a room of type z is 500.

On a pleasant night, the individual's surplus from a room of type x is 100, from a room of type y it is 500 and from a room of type z is 1000.

It is easy to see that the weather-dependent surpluses are consistent with the weather-dependent rankings.

The expected surplus from a room of type x is $533\frac{1}{3}$, the expected surplus from a room of type y is $366\frac{2}{3}$ and the expected surplus from a type of room z is $-666\frac{2}{3}$.

Suppose the individual is "risk neutral".

Thus, had this information been available to the individual at the time of booking, the individual would have chosen x.

Hence the indecisiveness noticed earlier is an instance of "bounded rationality due to lack of sufficient information".

In a private contribution, Professor Prasanta Pattanaik suggested that bounded rationality could arise out of a much greater informational deficit, i.e., lack of information about the probabilities of the states of nature. Clearly, this would be a very general starting point for investigating the consequences of "bounded rationality due to lack of sufficient information".

In a different context, Professor Pattanaik mentioned his work in Pattanaik (1968), which seems to be related to what we are discussing here. I wish to thank him for showing me the way to Pattanaik (1968). In Pattanaik (1968), the problem faced by an individual is to choose one from a non-empty finite set X of societies that the individual could migrate to. Let #X denote the cardinality (i.e., the number of societies) in set X. For each society in X, there are 'n' possible positions that the individual may end up being placed in, resulting in a state of nature $s \in S = \{1, \ldots, n\} \times X$. A typical state of nature, (j, x), represents the event "the individual chooses society 'x' and is assigned position j". Pattanaik (1968) assumes the "first best" situation, where the individual's preferences are represented by a **state-dependent utility function** $u: X \times S \to \mathbb{R}$ satisfying the property that for all $x \in X$ and $(j, x) \in S$, $u(x, (j, y)) = 0$ if $x \neq y$. Since, from the perspective of "nature" − the hypothetical entity that chooses or assigns the position to the individual − a priori, the probability of each society being chosen is the same as that of any other, the admissible set of subjective probability distributions over the states of nature S is a function of the form $p:\{1,\ldots,n\}\times X\to[0,1]$ such that for each $x \in X$, $p(j, x) = \frac{q(j|x)}{\#X}$, where for each $x \in X$ and $j \in \{1, \ldots, n\}$, $q(j|x) \in [0,1]$ and $\sum_{j=1}^{n} q(j|x) = 1$. Here q(j|x), may be interpreted as the probability of the event of being assigned the j^{th} position conditional on migrating to society 'a'. The probability distribution 'p' is the individual's assessment of the randomized strategy chosen by nature.

Note that $\sum_{s\in S} p(s) = \sum_{x\in X}(\sum_{j=1}^{n} \frac{q(j|x)}{\#X}) = \sum_{x\in X}(\frac{1}{\#X}\sum_{j=1}^{n} q(j|x)) = \sum_{x\in X}\frac{1}{\#X} = 1$.

Given such a 'p', the individual's problem is to choose an $x \in X$ that maximizes $\sum_{s\in S} u(x,s)p(s)$ which is equivalent to choosing an $x \in X$ that maximizes $\sum_{j=1}^{n} u(x,(j,x))q(j|x)$.

The purpose of our example in this section is to point out that the information required to formulate individual preferences in terms of state-dependent utility functions may be difficult to access and at best one may have state-dependent preferences represented by a “partial preference relation”, leading to “bounded rationality” that may be consistent with optimization and yet lead to sub-optimal outcomes, simply due to insufficient information.

I am also very grateful to Itzhak Gilboa for his informed observation which includes the following observations (which he honestly claims to be his personal views on “bounded rationality”):

(a) State-dependent preferences or utilities are not a reflection of bounded rationality.

(b) “(…) the term «bounded rationality» was coined by Simon, who had something much more dramatic in mind than what most people refer to by the term since he wanted to reject the entire optimization paradigm, replacing it by satisficing. While satisficing can also be embedded in the optimization framework, at least formally, it seems to me more of a deviation from the classical paradigm than, say, bounded memory, non-material payoffs and other models that explicitly are about optimization of something”.

Given that expected surplus maximization is simply expected utility maximization by a risk-neutral individual, we have no reason to disagree with his claim that representation of preferences by state-dependent utility functions do not imply “bounded rationality”. However, the example in this note clearly shows that state-dependent preferences may not always be able to perform the same “decision aiding” tasks that state-dependent utility functions are able to perform and so we would hesitate to treat the two concepts at par, at least in the context of decision aiding/making under uncertainty. Furthermore, the observation in (b) is a statement of fact, very correctly and succinctly expressed for the benefit of those like us, who may have limited knowledge of “mainstream bounded rationality” − theory and applications. Note that unlike the received theory of bounded rationality originating in the work of Herbert Simon, we focus on “lack of sufficient information” and not on “computational complexity” as our major concern for not being able to perform optimal decision making. Not being able to perform optimal decision making need not necessarily imply that the decision maker is not solving an optimization problem, as is mentioned in the last sentence of (b) above. The existing literature on bounded rationality focuses on behavioral issues related to computational constraints and complexity, which prevents individuals from solving optimization problems. We focus on the problems arising in “decision aiding” − the kind that technically qualified consultants may face − due to absence of sufficient information. Hence, although the decision aiding process involves optimization, the outcome of the process may turn out to be suboptimal, simply due to lack of available information.

Acknowledgment

Work on this paper began when I was with the School of Petroleum Management, PD Energy University (till June 5, 2022). I would like to thank Karl Schlag for comments on an earlier paper of which this paper is a considerably revised extension. The earlier paper was presented at the "Virtual Conference on Social Choice Theory and Organizations" on February 6, 2021, organized by Jac Heckelman (Wake Forest University, US), where I befitted immensely from comments of both Alexander Karpov and Elizabeth Maggie Penn. Those comments were an important reason for the revised extension discussed here. Thanks a lot, to all concerned, for their valuable inputs leading to this paper. I would also like to gratefully acknowledge comments on this paper and information about related research received from Itzhak Gilboa. His views about the solution concept available in this paper proved very useful in updating it. I would also like to thank Subhadip Chakraborty for valuable comments and suggesting corrections in the paper. As always, I wish to thank Professor Tadeusz Trzaskalik for his helpful comments and in particular for endorsing my attempts to re-orient "MCDM with weights". Finally, I wish to extend a very warm "thank you" to two very knowledgeable anonymous referees of this paper, particularly the one who forced me to take a fresh look at what is now a nonexistent and erroneous third theorem in a currently nonexistent section of the paper. Each and every comment from them was to say the least "very enlightening".

References

Campbell D., Kelly J., Qi S. (2018), *A Stability Property in Social Choice Theory*, International Journal of Economic Theory, 14, 85-95.

Congar R., Merlin V. (2012), *A Characterization of the Max-min Rule in the Context of Voting*, Theory and Decision, 72(1), 131-147.

Denicolo V. (1985), *Independent Social Choice Correspondences Are Dictatorial*, Economics Letters, 19(1), 9-12.

Endriss U. (2011), *Logic and Social Choice Theory* [in:] A. Gupta, J. van Benthem (eds.), *Logic and Philosophy Today*, College Publications.

Gilboa I. (1988), *A Combination of Expected Utility and Max-min Decision Criteria*, Journal of Mathematical Psychology, 32(4), 405-420.

Gilboa I., Schmeidler D. (1989), *Max-min Expected Utility with Non-Unique Prior*, Journal of Mathematical Economics, 18, 141-153.

Karni E. (1985), *Decision Making under Uncertainty: The Case of State-Dependent Preference*, Harvard University Press.

Kelsey D., Quiggin J. (1992), *Theories of Choice under Ignorance and Uncertainty*, Journal of Economic Surveys, 6(2), 133-153.

Lahiri S. (2020), *Generalized Sen-Coherence and Existence of Preferred with Probability at Least Half Winners*", International Journal of Operations Research, 17(3), 93-99.

Lahiri S. (2020/2021), *Extended Choice Correspondences & an Axiomatic Characterization of the Probabilistic Borda Rule*, Mathematical Methods in Economics and Finance − m^2ef, 16/17(1), 15-34.

Lahiri S. (2021), *Pattanaik's Axioms & the Existence of Winners Preferred with Probability at Least Half*, Operations Research & Decisions, 31(2), 109-122.

Maskin E. (1979), *Decision-Making under Ignorance with Implications for Social Choice*, Theory and Decision, 11, 319-337.

Pattanaik P.K. (1970), *Sufficient Conditions for the Existence of a Choice Set under Majority Voting*, Econometrica, 38(1), 165-170.

Pattanaik P.K. (1968), *Risk, Impersonality and Social Welfare Functions*, Journal of Political Economy, 76(6), 1152-1169.

Puppe C., Schlag K. (2009), *Choice under Complete Uncertainty when Outcome Spaces Are State Dependent*, Theory and Decision, 66(1), 1-16.