ON THE TOLERANCE WAVE - TYPE SOLUTIONS TO THE HYPERBOLIC HEAT CONDUCTION IN MICROPERIODIC COMPOSITES

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Abstract. In the note wave-type solutions to the hyperbolic heat conduction in the microperiodic composites are investigated. The considerations are related to the tolerance averaged model of the hyperbolic heat transfer problems in microperiodic composites. The Lapunov exponent notation is applied. The open form of these solutions are formulated in one shape function case.

Introduction

Many physical problems can be described by the Cauchy problem for a system of ordinary differential equations with constant coefficients, which in the consistent matrix form can be written in the form

$$\dot{x}^T = Ax^T,\ x(0+) = x_o \tag{1}$$

for known real matrices A and x_o of $n\times n$ and $1\times n$ dimensions, respectively, and unknown $1\times n$ matrix $x = x(t)$, t > 0. The solution to this equation can be written in the form

$$x^T = e^{At}{x_o}^T \tag{2}$$

where

$$e^{At} = \sum_{n=0}^{\infty}\frac{A^n t^n}{n!} \tag{3}$$

is the Lapunov exponential of matrix *At.* It is the well known fact that

$$e^{At}e^{Bt} = e^{(A+B)t} \tag{4}$$

provided that $AB = BA$. Indeed if $x^T = e^{-Bt}y^T$ then and under (2) and ${y_o}^T = e^{-Bt}{x_o}^T$ one can obtain

$$y^T = e^{Bt} e^{At} y_o{}^T \tag{5}$$

and

$$\dot{y} = (e^{Bt} A e^{-Bt} + B) y \tag{6}$$

Hence under assumption $AB = BA$ and (6)

$$y^T = e^{(A+B)t} y_o{}^T \tag{7}$$

and then conditions (5), (7) yield (4).

In this note we are to apply mentioned above Lapunov exponents notation properties to the discussion of the wave-type solutions in the hyperbolic heat conduction in the microperiodic solid.

1. Formulation of the problem

The starting point of consideration will be tolerance averaged model of the hyperbolic heat conduction in the one dimensional microperiodic solid which occupy the interval $[0,L]$ in the reference configuration. Denoting by c mean heat, by τ relaxation time and by k conductivity constant and assuming that the total temperature together with temperature gradient in every constituent for a certain shape functions sequence $h = [h^1,\dots, h^n]$ can be approximated with a sufficient accuracy by decompositions

$$\theta(x,t) = u(x,t) + h(x)\upsilon^T(x,t),\ \theta_x(x,t) = u_x(x,t) + h_x(x)\upsilon^T(x,t) \tag{8}$$

respectively. Here and in the subsequent considerations symbol $(\cdot)_x$, $(\cdot)_{xx}$ denote spatial derivatives. In above decompositions $u = u(x)$ and $\upsilon = \upsilon(x) = [\upsilon^1,\dots, \upsilon^n]$ are new basic unknowns named as the averaged temperature and the vector of temperature amplitudes. Temperature introducing a certain finite which will be rewritten here in the form, cf. [1]

$$\begin{aligned} &\langle c_\tau\rangle \ddot{u} + \langle c\rangle \dot{u} - \langle k\rangle u_{xx} - [k]\upsilon^T{}_x = 0 \\ &\gamma_\tau \ddot{\upsilon} + \gamma\dot{\upsilon} + \{k\}\upsilon^T + [k]^T u_x = 0 \end{aligned} \tag{9}$$

where under the averaged operator over the unit cell $[-\lambda/2,\lambda/2]$

$$\langle f\rangle(x) = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(y)dy, \quad x \in [\lambda/2, L-\lambda/2] \tag{10}$$

matrix coefficients are given by

$$\gamma = \begin{bmatrix} \langle ch^1h^1\rangle & \dots & \langle ch^1h^n\rangle \\ \dots & \dots & \dots \\ \langle ch^nh^1\rangle & \dots & \langle ch^nh^n\rangle \end{bmatrix}, \quad \gamma_\tau = \begin{bmatrix} \langle c_\tau h^1h^1\rangle & \dots & \langle c_\tau h^1h^n\rangle \\ \dots & \dots & \dots \\ \langle c_\tau h^nh^1\rangle & \dots & \langle c_\tau h^nh^n\rangle \end{bmatrix}$$

$$\{k\} = \begin{bmatrix} \langle kh_x^1h_x^1\rangle & \dots & \langle kh_x^1h_x^n\rangle \\ \dots & \dots & \dots \\ \langle kh_x^nh_x^1\rangle & \dots & \langle kh_x^nh_x^n\rangle \end{bmatrix}, \quad [k] = \left[\langle kh_x^1\rangle, \dots, \langle kh_x^n\rangle\right] \tag{11}$$

We are to investigate wave-type solutions of the tolerance equations (9) of the form

$$u(x,t) = F(x-\mu t), \quad \upsilon^T(x,t) = G(x-\mu t) \tag{12}$$

where $F(\cdot)$ and $G(\cdot)$ are new unknowns. Similar problem has been investigated in [1]. Under (12) tolerance model equations (9) take form

$$\begin{aligned} &(\langle c_\tau\rangle\mu^2 - \langle k\rangle)F'' - \langle c\rangle\mu F' - [k]G' = 0 \\ &\gamma_\tau\mu^2 G'' + ([k]^T - \gamma\mu)G' + \{k\}G = 0 \end{aligned} \tag{13}$$

where prime denotes the derivation of the wave-type generators *F* and *G*. Introducing additional unknowns $U(\cdot)$ and $V(\cdot)$ by interrelations

$$U(y) = F'(y), \; V(y) = G'(y) \tag{14}$$

and denoting unit $n \times n$ matrix by I_n the equations (13) can be rewritten in matrix form

$$\begin{bmatrix} \langle c_\tau\rangle\mu^2 - \langle k\rangle & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & \gamma_\tau\mu^2 \end{bmatrix} \begin{bmatrix} U' \\ G' \\ V' \end{bmatrix} = \begin{bmatrix} \langle c\rangle\mu & 0 & [k] \\ 0 & 0 & I_n \\ 0 & -\{k\} & -([k]^T - \gamma\mu) \end{bmatrix} \begin{bmatrix} U \\ G \\ V \end{bmatrix} \tag{15}$$

which, together with a certain initial conditions and under condition $U = F'$, is equivalent to the Cauchy problem (1) for

$$A=\begin{bmatrix} \dfrac{\langle c\rangle\mu}{\langle c_\tau\rangle\mu^2-\langle k\rangle} & 0 & [k] \\ 0 & 0 & I_n \\ 0 & -\dfrac{1}{\mu^2}\gamma_\tau^{-1}\{k\} & -\dfrac{1}{\mu^2}\gamma_\tau^{-1}([k]^T-\gamma\mu) \end{bmatrix} \tag{16}$$

$$x_0=[U_0,G_0,V_0]^T$$

The problem we are to discuss the solution to the Cauchy problem (1) specified by (16).

2. Analysis

To discuss solutions to the Cauchy problem (1) specified by (16) we are to decompose matrix A given in (16) into sum

$$A=H+\frac{1}{2n+1}trA\ \ I_{2n+1} \tag{17}$$

and hence

$$H=\begin{bmatrix} \dfrac{\langle c\rangle\mu}{\langle c_\tau\rangle\mu^2-\langle k\rangle}-\dfrac{1}{2n+1}trA & 0 & [k] \\ 0 & -\dfrac{1}{2n+1}trA\ \ I_n & I_n \\ 0 & -\dfrac{1}{\mu^2}\gamma_\tau^{-1}\{k\} & -\dfrac{1}{\mu^2}\gamma_\tau^{-1}([k]^T-\gamma\mu)-\dfrac{1}{2n+1}trA\ \ I_n \end{bmatrix} \tag{18}$$

Since terms in decomposition (17) commute solution to the mentioned above Cauchy problem can be written in the form

$$\begin{bmatrix} U \\ G \\ V \end{bmatrix}=e^{Hy}e^{trAy}x_0 \tag{19}$$

It must be emphasized that the term e^{trAy} of the right hand side of (19) is a scalar one. At the same time H a certain traceless matrix coefficient, $J_1 = trH = 0$. Hence the Caqyley-Hamilton polynomial related to this matrix has the form

$$z^{2n+1}+J_2z^{2n-1}+J_3z^{2n-1}+\ldots+J_{2n}z+\det H \tag{20}$$

where $J_1, J_2, ...J_{2n}, J_{2n+1} = \det H$ are invariants of *H*. It is mean that term $trHz^{2n}$ in (20) is equal to zero.

Now we are to restrict considerations to the special case in which the tolerance model of the hyperbolic heat conduction includes exclusively one shape function. In this case matrix *A* as well as *H* are of 3×3 dimension and hence polynomial (20) takes the form

$$z^3 + J_2 z + \det H \tag{21}$$

It is mean that (21) has three roots of the form *a*, –0.5*a*+*bi*, –0.5*a*–*bi* and hence

$$e^{Hy} = O\begin{bmatrix} e^{ay} & 0 & 0 \\ 0 & e^{-ay/2}(\cos by + i\sin by) & 0 \\ 0 & 0 & e^{-ay/2}(\cos by - i\sin by) \end{bmatrix} O^T \tag{22}$$

for a certain orthogonal matrix *O*.

More detailed analysis and the physical reliability discussion of the solution (19) to the considered Cauchy problem will be explained elsewhere.

References

[1] Woźniak C., Woźniak M., 2D-dynamics of a stratified elastic subsoil layer, Arch. Appl. Mech. 1996, 66, 284-290.

[2] Moller C., Van Loan C., Nineteen dubious ways to compute the exponential of the matrix. Twenty-five years later, SIAM Review, Society for Industrial and Applied Mathematics 2003, 74, 1, 3000.