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ON AXIOMS OF CONVERGENCE IN LINEAR SPACES

Abstract. By a (general) convergence in a given linear space X we mean a mapping $G: X^N \to 2^X$, where N denotes the set of all positive integers, and by a zero-convergence in X we mean a convergence G_0 in X for which $G_0(x) \neq \emptyset$ implies $0 \in G_0(x)$ for each $x \in X^N$. In the paper, the two operations are defined: 1° operation C, which to each zero-convergence G_0 in X assigns some general convergence G in X, and 2° operation C_0, which to each general convergence G in X assigns a zero-convergence G_0 in X. Various systems of axioms for general convergences and zero-convergences are considered and their connections with the operations C and C_0 are studied. Also mutual independence of axioms is studied.

Very often convergences are defined by topology. However there exist important convergences which cannot be defined in this way, e.g., type I and type II convergences in the Mikusiński operational calculus (see [9], [2], [3]).

These and other examples show the need of development of a generel theory, in which convergence in a given space is defined immediately by indicating convergent sequences and their limits and some general conditions (axioms) are supposed (see e.g. [10]).

Of course, such a convergence can be treated as a function, which to every sequence assigns a set of limits (the empty set if a sequence is divergent; a one-element set if a convergent sequence has a unit limit).

In particular, topological convergence (i.e. convergence defined by some topology) can be characterized in terms of conditions mentioned above. For Hausdorff convergences it is done in [7] and [8] and for multivalued convergences (i.e. without the assumption of uniqueness) in the paper [4] (see also [5] and [6]).

One can consider convergences in spaces equipped with some algebraic structure, e.g., in groups or in linear spaces (see e.g. [11]). In linear

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spaces only the sequences convergent to 0 are usually defined (zero-convergence). All the convergent sequences (general convergence) can be then defined by linearity.

In this note we present a general scheme how to pass in linear spaces from the definition of sequences convergent to 0 to the definition of sequences convergent to arbitrary elements and conversely.

More precisely, we first introduce axiomatically two kinds of convergence in a given linear space: zero-convergence and general convergence. Next we define an operation C assigning to every zero-convergence a general convergence and an operation C_0, which makes correspond to every general convergence a zero-convergence (section 1). We study what axioms are preserved when operations C and C_0 are performed (section 2).

In turn, we discuss relations between operations C and C_0 (section 3). In particular, we find conditions, under which the identities $CC_0G = G$ and $C_0CG_0 = G_0$ hold for any general convergence G and zero-convergence G_0.

Finally, we discuss independence of axioms (section 4).

1. We shal denote: by N — the set of all positive integers, by R — the set of all real numbers, by X — an arbitrary fixed set, by E — a fixed linear space over the field R, by Greek letters $\xi, \eta, \ldots$ — elements of X or E, by Latin letters $x, y, \ldots$ — elements of X^N or E^N, i.e. sequences $\{\xi_n\}$, $\{\eta_n\}, \ldots$ of elements of X or E respectively.

If y is a subsequence of a sequence x, then we shall write $y \prec x$; the constant sequence $\xi, \xi, \xi, \ldots$, where $\xi \in X$ (or $\xi \in E$), will be denoted by $\dot{\xi}$ and the set $\{\xi_n : n \in N\}$ for $x = \{\xi_n\}$ — by (x) (cf. notation in [1]).

If A, $B \subset E$ and $\lambda \in R$, then we shall use the standard notation: $A + B = \{x + y : x \in A, y \in B\}$, $\lambda A = \{\lambda x : x \in A\}$ and the convention: $A + \emptyset = \emptyset + A = \emptyset$, $\lambda\emptyset = \emptyset$.

By a *general convergence* (shortly: *convergence*) *on a given set* X, we mean a mapping from X^N into 2^X (cf. [1]).

Let G and G' be two convergences on X. We write $G \subset G'$ if $G(x) \subset G'(x)$ for every $x \in X^N$. If $G \subset G'$ and $G' \subset G$, then we write $G = G'$.

The following axioms concerning a convergence G on X were considered in [7], [8], [1], [9], [4]—[6]:

F. If $y \prec x$, then $G(x) \subset G(y)$;

U. If $\xi \notin G(x)$, then there exists $y \prec x$ such that $\xi \notin G(z)$ for every $z \prec y$;

H. For every $x \in X^N$ the set $G(x)$ contains at most one element;

S. $\xi \in G(\dot{\xi})$ for every $\xi \in X$.

It is convenient to consider the following axiom, complementary with respect to axioms S and H:

S′. If $\eta \in G(\dot{\xi})$, then $\xi = \eta$.

In E, it is natural to consider for general convergences besides F, U, H, S, S' also the following axioms of linearity (cf. [10]):

A. $G(x)+G(y)\subset G(x+y)$, $(x, y\in E^N)$;

M. $\lambda G(x)\subset G(\lambda x)$, $(x\in E^N, \lambda\in R)$

or the following weaker versions of linearity:

T. If $\xi\in G(x)$, then $0\in(x-\dot{\xi})$, $(x\in E^N)$;

T' If $0\in G(x-\dot{\xi})$, then $\xi\in G(x)$, $(x\in E^N)$.

A (general) convergence G on a linear space E will be called a *zero–convergence* if

$$G_0(x)\neq\emptyset \text{ implies } 0\in G_0(x),\ (x\in E^N).$$

For zero-convergences, we consider the above axioms, too. However it seems to be more natural for those convergences to replace axioms S and S' by the weaker ones:

S_0. $0\in G_0(\dot{0})$ or, equivalently, $G_0(\dot{0})\neq\emptyset$.

S_0'. If $G_0(\dot{\xi})\neq\emptyset$ (i.e. $0\in G_0(\dot{\xi})$) for $\xi\in X$, then $\xi=0$.

Relations between the above axioms will be studied later.

Now, we are going to introduce the following operations: 1° operation C assigning a general convergence G on E to every zero-convergence G_0 on E; 2° operation C_0 assigning a zero-convergence G_0 on E to every general convergence G on E.

Namely, for a given zero-convergence G_0 on E we define the general convergence $CG_0=G$ as follows:

$$\xi\in G(x)\Leftrightarrow 0\in G_0(x-\dot{\xi}),\ (x\in E^N).$$

For a given general convergence G on E we construct the zero-convergence $C_0G=G_0$ on E in the following way:

$$G_0(x)=G(x) \text{ if } 0\in G(x) \text{ and } G(x)=\emptyset \text{ otherwise.}$$

Note that the above definition of the operation C is based on linearity, but the definition of C_0 is not. Therefore it seems to be reasonable, for a given general convergence G, to treat a sequence x as convergent to 0 in the sense of a zero-convergence whenever for some $\eta\in R$ we have $\eta\in G(x+\dot{\eta})$. Accordingly, we define the second version of operation of type 2° in the following way:

$$\bar{C}_0G(x)=\bar{G}_0(x)=\bigcup\{G(x+\dot{\eta})-\eta\},\ (x\in E^N) \tag{1.1}$$

for the given general convergence G, where the union is taken over such $\eta\in X$ that $\eta\in G(x+\dot{\eta})$; if for some $x\in E^N$ such η does not exist, then we adopt $\bar{G}_0(x)=\emptyset$.

It is obvious that $\bar{G}_0$ is a zero-convergence on E.

PROPOSITION 1.1. *For every general convergence G, we have*

$$C_0G\subset\bar{C}_0G. \tag{1.2}$$

If G satisfies axioms S *and* A, *then*

$$C_0G = \bar{C}_0G. \tag{1.3}$$

P r o o f. Let x be arbitrary. If $\xi \in C_0G(x)$, then $\xi, 0 \in C_0G(x) = G(x)$ and, by (1.1),

$$\xi \in G(x+\dot{0})-0 \subset \bar{C}_0G(x),$$

i.e., (1.2) holds.

Now, let $\xi \in \bar{C}_0G(x)$. Then there exists $\eta \in X$ such that $\eta, \xi + \eta \in G(x+\dot{\eta})$. Hence, by S and A, we have

$$0 = \eta-\eta \in G(x+\dot{\eta})+G(-\dot{\eta}) \subset G(x)$$

and

$$\xi = \xi+\eta-\eta \in G(x+\dot{\eta})+G(-\dot{\eta}) \subset G(x).$$

But this means that $\xi \in G(x) = C_0G(x)$ and thus the second inclusion of (1.3) is true under S and A.

EXAMPLE 1.1. Let $E = R$. We define $G(x) = \Phi$ if the sequence $x = \{\xi_n\}$ is not convergent in the usual sense and $G(x) = \{-\xi\}$ if $\lim_{n\to\infty} \xi_n = \xi$ in the usual sense.

Note that G fulfils A (and U, which will be needed later), but does not fulfil S.

If $\xi_n \to \xi \neq 0$, we have $C_0G(x) = \Phi$. On the other hand, it is easy to see that

$$\bar{C}_0G(x) = G\left(x-\frac{\dot{\xi}}{2}\right)+\frac{\xi}{2} = \{0\},$$

i.e., (1.3) does not hold.

EXAMPLE 1.2. Let $E = R$ and $x = \{\xi_n\}$. If

$$\xi_n \searrow \xi > 0, \text{ i.e., } \xi_n \to \xi \text{ and } \xi_n \geqslant \xi \text{ for almost all } n,$$

or if

$$\xi_n \nearrow \xi < 0, \text{ i.e. } \xi_n \to \xi \text{ and } \xi_n \leqslant \xi \text{ for almost all } n,$$

then we adopt $G(x) = \{\xi\}$. Moreover $G(x) = \{0\}$ if $x = \{\xi_n\}$ and $\xi_n = 0$ for almost all n. In the remaining cases, we put $G(x) = \Phi$.

Note that G satisfies S (and U, which will be needed later) but does not satisfy A.

For $x = \left\{\frac{1}{n}\right\}$ and $x' = \left\{-\frac{1}{n}\right\}$ we have

$$C_0G(x) = C_0G(x') = \Phi$$

and

$$\bar{C}_0G(x) = \bigcup_{\eta>0}\left[G\left(\left\{\eta+\frac{1}{n}\right\}\right)-\eta\right] = \{0\},$$

$$\bar{C}_0 G(x') = \bigcup_{\eta > 0} \left[G\left(\left\{\eta - \frac{1}{n}\right\}\right) - \eta \right] = \{0\},$$

i.e., (1.3) does not hold.

2. In this section we study if individual axioms are preserved under the operations C, C_0 and $\bar{C}_0$.

PROPOSITION 2.1. *If a zero-convergence G_0 satisfies* F, *then the general convergence $G = C\,G_0$ satisfies* F. *If a general convergence G satisfies F, then the zero-convergences. $G_0 = C_0\,G$ and $\bar{G}_0 = \bar{C}_0\,G$ satisfy* F.

P r o o f. Suppose that G_0 satisfies F and let $y \prec x$ and $\xi \in G(x)$. By the definition of G, we have $0 \in G_0(x - \dot{\xi})$. Since G_0 fulfils F, we have $0 \in G_0(y - \dot{\xi})$ and hence $\xi \in G(y)$. Thus we have proved that $G(x) \subset G(y)$, i.e., G satisfies F.

Suppose now that G fulfils F and let $y \prec x$. If $0 \notin G(y)$, then $0 \notin G(x)$, by F. Hence $G_0(x) = \emptyset \subset G_0(y) = \emptyset$. If $0 \in G(y)$, then $G_0(y) = G(y) \supset G(x) \supset G_0(x)$, because F holds for G. Thus we have proved that G_0 fulfils F.

Since F is assumed for G, we have

$$G(x + \dot{\eta}) \subset G(y + \dot{\eta}) \tag{2.1}$$

and

$$G(x + \dot{\eta}) - \eta \subset G(y + \dot{\eta}) - \eta \tag{2.2}$$

for each $\eta \in E$. In view of the definition of $\bar{G}_0$, inclusions (2.1) and (2.2) yield $\bar{G}_0(x) \subset \bar{G}_0(y)$, i.e., $\bar{G}_0$ fulfils F and the proof is finished.

PROPOSITION 2.2. *If a zero-convergence G_0 satisfies* U, *then the general convergence $G = CG_0$ satisfies* U. *If a general convergence G satisfies* U, *then the zero-convergence $G_0 = C_0G$ satisfies U. If a general convergence G satisfies* U, S *and* A, *then the zero-convergence $\bar{G}_0 = \bar{C}_0G$ satisfies* U.

P r o o f. Suppose U for G_0 and let $\xi \notin G(x)$, i.e., $0 \notin G_0(x - \dot{\xi})$. Then there exists $y \prec x$ such that $0 \notin G_0(z - \dot{\xi})$ for each $z \prec y$. This means, $\xi \notin G(z)$ for each $z \prec y$, i.e., G fulfils U.

Suppose now that G fulfils U and let for each subsequence y of a given sequence x exist $z \prec y$ such that $\xi \in G_0(z)$. But then $G_0(z) = G(z)$ and $0, \xi \in G(z)$. Hence, by U, we have $0, \xi \in G(x) = G_0(x)$. This proves the second part of the proposition.

The last part follows from the second one, by Proposition 1.1.

The assumption in the third part of Proposition 2.2 that G satisfies axioms S and A cannot be omitted.

In fact, the convergence G from Example 1.1 fulfils U and A, but does not fulfil S. We have in this case $\bar{G}_0(x) = \bar{C}_0G(x) = \{0\}$, provided there is a $\xi \in R$ such that $\xi_n \to \xi$, and $\bar{G}_0(x) = \emptyset$ otherwise. The zero-

-convergence $\bar{G}_0$ does not fulfil U, because $0 \notin \bar{G}_0(x)$ for $x = (1, -1, 1, -1, \ldots)$, but for each subsequence y of x there exists subsequence z of y, which is of the form $z = (1, 1, \ldots)$ or $z = (-1, -1, \ldots)$, i.e., $0 \in \bar{G}_0(z)$.

On the other hand, the convergence G from Example 1.2 satisfies U and S, but does not fulfil A. The zero-convergence $\bar{G}_0 = \bar{C}_0 G$ does not fulfil axiom U, because $0 \notin \bar{G}_0(x)$ for $x = \left(1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots\right)$, but for each subsequence y of x there exists a subsequence z of y, which is either a subsequence of the sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ or of the sequence $\left(-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\right)$, i.e., $0 \in \bar{G}_0(z)$.

Now, note that axiom H is not preserved in general when the operation C is applied, as examples below will show. However we have the following statement.

PROPOSITION 2.3. *If a zero-convergence* G_0 *satisfies axioms* A, M *and* S_0', *then the general convergence* $G = CG_0$ *satisfies* H. *If a general convergence satisfies* H, *then the zero-convergences* $G_0 = C_0 G$ *and* $\bar{G}_0 = \bar{C}_0 G$ *satisfy* H.

P r o o f. Suppose that a zero-convergence G_0 fulfils A, M and S_0' and let $\xi, \eta \in G(x)$. By the definition of G, we have $0 \in G_0(x - \dot{\xi})$ and $0 \in G_0(x - \dot{\eta})$. Hence, by M and A, we get

$$0 \in G_0(x - \dot{\xi}) + G(\dot{\eta} - x) \subset G(\dot{\eta} - \dot{\xi})$$

whence $\xi = \eta$ results, by virtue of S_0'. The first part of the proposition is shown.

Assume now that a general convergence G fulfils H. If $\xi \in G_0(x)$, then $0, \xi \in G(x) = G_0(x)$ and, since G satisfies H, we get $\xi = 0$, i.e., the zero-convergence G_0 satisfies H. If $\xi \in \bar{G}_0(x)$, then by the definition of $\bar{G}_0$ there exists $\eta \in E$ such that $\eta, \xi + \eta \in G(x + \dot{\eta})$ and hence $\eta = \xi + \eta$, i.e., $\xi = 0$. Thus the zero-convergence $\bar{G}_0$ satisfies H too and the proof is completed.

EXAMPLE 2.1. Let $E = R$ and let $G_0(x) = \{0\}$ if $x = \dot{\xi}$ for $\xi \in R$ and $G_0(x) = \emptyset$ otherwise. Of course, the zero-convergence G fulfils H, A, M, and axiom S_0' is not fulfilled. The general convergence $G = CG_0$ does not satisfy H, because $G(\dot{\xi}) = R$ for every $\xi \in R$.

EXAMPLE 2.2. Let $E = R$. Define the zero-convergence G_0 as follows: $G_0(\dot{0}) = \{0\}$, $G_0(\{\xi + a_n\}) = \{0\}$ for any $\xi \in R$ and $a_n \to 0$ with $a_n \neq 0$ $(n \in N)$; in remain cases let $G_0(x) = \emptyset$.

Evidently, the zero-convergence G_0 fulfils axioms H, M, S_0'. Axiom A is not satisfied, because

$$G_0\left(\left\{\xi + \frac{1}{n}\right\}\right) + G_0\left(\left\{\eta - \frac{1}{n}\right\}\right) = \{0\}$$

and

$$G_0(\dot{\xi}+\dot{\eta}) = \Phi$$

for any $\xi, \eta \in R$ such that $\xi+\eta \neq 0$. Note that $G\left(\left\{\xi+\frac{1}{n}\right\}\right) = R$, i.e., axiom H does not hold for $G = CG_0$.

EXAMPLE 2.3. Let $E = R$. We adopt $G_0(x) = \{0\}$ if $x = 0$ or if $x = \{\xi_n\}$, where $\xi_n \to \infty$; otherwise let $G_0(x) = \Phi$.

It is clear that the zero-convergence G_0 satisfies axioms H, A and S_0' and does not fulfil axiom M. Let us note that $G(x) = CG_0(x) = R$, i.e. the general convergence G has not property H.

PROPOSITION 2.4. *If a zero-convergence* G_0 *satisfies* S_0, *then the general convergence* $G = CG_0$ *satisfies* S. *If a general convergence* G *satisfies* S, *then the zero-convergences* $G_0 = C_0G$ *and* $\bar{G}_0 = \bar{C}_0G$ *satisfy* S_0.

P r o o f. The first part follows from the fact that the condition $\xi \in G(\dot{\xi})$ is equivalent, by the definition of G, to the condition $0 \in G_0(\dot{0})$.

If G satisfies S, then we have in particular $0 \in G(\dot{0}) = G_0(\dot{0})$, that means G_0 satisfies S_0. On the other hand, we have then $\eta \in G(\dot{\eta})$ and $0 \in G(\dot{\eta})-\eta$ for every $\eta \in E$, i.e., $0 \in \bar{G}_0(\dot{0})$.

PROPOSITION 2.5. *If a zero-convergence* G_0 *satisfies* S_0', *then the general convergence* $G = CG_0$ *satisfies* S'. *If a general convergence* G *satisfies* S', *then the zero-convergence* $G_0 = C_0G$ *and* $\bar{G}_0 = \bar{C}_0G$ *satisfy* S_0'.

P r o o f. Assume that G_0 fulfils S_0' and let $\eta \in G(\dot{\xi}) = CG_0(\dot{\xi})$. This means $0 \in G_0(\dot{\xi}-\dot{\eta})$ and thus $\xi = \eta$, by S_0', which shows the first part of our assertion.

Now, let G fulfil S' and let $0 \in G_0(\dot{\xi}) = C_0G(\dot{\xi})$. This means $0 \in G(\dot{\xi}) = G_0(\dot{\xi})$ and $\xi = 0$, by S'.

In turn, if $0 \in \bar{G}_0(\dot{\xi}) = \bar{C}_0G(\dot{\xi})$, then by the definition of $\bar{G}_0$ there exists $\eta \in E$ such that $\eta \in G(\dot{\xi}+\dot{\eta})$. Hence, by virtue of S', we get $\xi+\eta = \eta$, which yields the desired assertion.

PROPOSITION 2.6. *If a zero-convergence* G_0 *satisfies* A, *then the general convergence* $G = CG_0$ *satisfies* A. *If a general convergence* G *satisfies* A, *then the zero-convergences* $G_0 = C_0G$ *and* $\bar{G}_0 = \bar{C}_0G$ *satisfy* A.

P r o o f. Assume that a zero-convergence G_0 fulfils A and that $\zeta \in G(x)+G(y)$, i.e. $\zeta = \xi+\eta$ with $\xi \in G(x)$ and $\eta \in G(y)$. By the definition of G, we have $0 \in G_0(x-\dot{\xi})$ and $0 \in G_0(y-\dot{\eta})$. Since A holds for G_0, we obtain

$$G_0(x-\dot{\xi})+G_0(y-\dot{\eta}) \subset G(x+y-(\dot{\xi}+\dot{\eta})),$$

i.e.,

$$\xi+\eta \in G(x+y),$$

which completes the proof of the first part.

Suppose now that a general convergence G fulfils A. If $0 \notin G(x)$ or $0 \notin G(y)$, then $G_0(x) = \emptyset$ or $G_0(y) = \emptyset$ respectively and, consequently,

$$G_0(x)+G_0(y) = \emptyset \subset G_0(x+y).$$

If $0 \in G(x)$ and $0 \in G(y)$, then

$$0 \in G(x)+G(y) \subset G(x+y),$$

since A holds for G, and thus

$$G_0(x)+G_0(y) = G(x)+G(y) \subset G(x+y) = G_0(x+y).$$

To prove the last assertion suppose that

$$\eta_1 \in G(x+\dot{\eta}_1) \text{ and } \eta_2 \in G(x+\dot{\eta}_2). \tag{2.3}$$

Of course we have

$$\eta_1+\eta_2 \in G(x+\dot{\eta}_1)+G(y+\dot{\eta}_2) \subset G(x+y+\dot{\eta}_1+\dot{\eta}_2)$$

and

$$[G(x+\dot{\eta}_1)-\eta_1]+[G(y+\dot{\eta}_2)-\eta_2] \subset G(x+y+\dot{\eta}_1+\dot{\eta}_2)-(\eta_1+\eta_2),$$

because A is satisfied by G. From this results the inclusion

$$G_0(x)+G_0(y) \subset G_0(x+y)$$

in the case, when there exist $\eta_1, \eta_2 \in E$ satisfying (2.3).

In the opposite case we have

$$G_0(x)+G_0(y) = \emptyset \subset G_0(x+y).$$

Thus the proof is complete.

PROPOSITION 2.7. *If a zero-convergence* G_0 *satisfies* M, *then the general convergence* $G = CG_0$ *satisfies* M. *If a general convergence satisfies* M, *then the zero-convergences* $G_0 = C_0G$ *and* $\bar{G}_0 = \bar{C}_0G$ *satisfy* M.

P r o o f. Assume that G_0 fulfils M and that $\zeta \in \lambda\, G(x)$ for $G = CG_0$, $\lambda \in R$ and $x \in E^N$. That means, we have $\zeta = \lambda\xi$ with $\xi \in G(x)$. Hence $0 \in G_0(x-\dot{\xi})$ and

$$0 \in \lambda G\,(x-\dot{\xi}) = \emptyset \subset G_0(\lambda(x-\dot{\xi})).$$

This yields, by the definition of G, the relation $\zeta = \lambda\xi \in G(\lambda x)$, which finishes the proof of the first part of the proposition.

In turn, assume that a general convergence G fulfils axiom M. If $0 \notin G(x)$, then

$$\lambda G_0(x) = \emptyset \subset G_0(\lambda x)$$

for every $\lambda \in R$ and $G_0 = C_0G$.

If $0 \in G(x)$, then $0 \in \lambda G(x) \subset G(\lambda x)$ and we get

$$\lambda G_0(x) = \lambda G(x) \subset G(\lambda x) = G_0(\lambda x),$$

owing to M holding for G.

It remains to prove that $\bar{G}_0 = \bar{C}_0 G$ fulfils M. If

$$\eta \in G(x+\dot{\eta}), \tag{2.4}$$

then we have, by virtue of M,

$$\mu = \lambda\eta \in \lambda G(x+\dot{\eta}) \subset G(\lambda x+\dot{\mu})$$

and

$$\lambda[G(x+\dot{\eta})-\eta] \subset G(\lambda x+\dot{\mu}) - \mu$$

for any $\lambda \in R$. This yields

$$\lambda \bar{G}_0(x) \subset \bar{G}_0(\lambda x), \quad (\lambda \in R)$$

in the case, when there exists $\eta \in E$ such that (2.4) holds.

In the converse case, we have

$$\bar{G}_0(x) = \emptyset \subset \bar{G}_0(\lambda x), \quad (\lambda \in R)$$

and the proof is over.

PROPOSITION 2.8. *The general convergence $G = CG_0$ satisfies axioms* T *and* T′ *for every zero-convergence G_0. If a general convergence G satisfies* T, *then the zero-convergences $G_0 = C_0 G$ and $\bar{G}_0 = \bar{C}_0 G$ satisfy* T.

P r o o f. The first assertion follows from the following equivalences:

$$\xi \in G(x) \Leftrightarrow G_0(x-\dot{\xi}) \Leftrightarrow 0 \in G(x-\dot{\xi}),$$

which are consequences of the definition of G.

To prove the second one suppose that G satisfies T. First let $\xi \in G_0(x) = C_0 G(x)$. Then $\xi, 0 \in G_0(x) = G(x)$ and, by T, we obtain $0 \in G(x-\dot{\xi})$, i.e.,

$$0 \in G_0(x-\dot{\xi}) = G(x-\dot{\xi}).$$

Finally note that the usual convergence G in R satisfies axiom T′ (and all others) and the zero-convergences $G_0 = C_0 G$ and $\bar{G}_0 = \bar{C}_0 G$ (which coincide with the usual convergence to 0 in R) do not fulfil it.

3. Now we are going to study connections between the operations C, C_0 and $\bar{C}_0$.

PROPOSITION 3.1. *If G, G' are general convergences on E and $G \subset G'$ then $C_0 G \subset C_0 G'$ and $\bar{C}_0 G \subset \bar{C}_0 G'$.*

P r o o f. First let a $\xi \in C_0 G(x)$. By the definition of C_0 this means that $\xi \in G(x)$ and $0 \in G(x)$. Hence, by the assumption, $\xi, 0 \in G'(x)$ and, consequently, $\xi \in C_0 G'(x) = G'(x)$.

Now let $\xi \in \bar{C}_0 G(x)$. This means that there exists an $\eta \in E$ such that $\eta, \xi+\eta \in G(x+\dot{\eta})$. By the assumption we have $\eta, \xi+\eta \in G'(x+\dot{\eta})$ and thus $\xi = \xi+\eta-\eta \in C_0 G'(x)$. The proof is complete.

PROPOSITION 3.2. *If G_0, G_0' are zero-convergences and $G_0 \subset G_0'$, then $CG_0 \subset CG_0'$.*

P r o o f. If $\xi \in CG_0(x)$, then $0 \in G_0(x-\dot{\xi})$ and, by the assumption, $0 \in G_0'(x-\dot{\xi})$. But this means that $\xi \in CG_0'(x)$, which completes the proof.

PROPOSITION 3.3. *If a general convergence G on E satisfies* S *and* **A**, *then*

$$CC_0G = G \tag{3.1}$$

and

$$C\bar{C}_0G = G. \tag{3.2}$$

P r o o f. In view of Proposition 1.1, it suffices to prove (3.1). Let $\xi \in G(x)$. By S, we have

$$0 = \xi - \xi \in G(x) + G(-\dot{\xi}).$$

Hence, by A, we get $0 \in G(x - \dot{\xi})$, i.e.,

$$0 \in C_0G(x - \dot{\xi}),$$

which means that $\xi \in CC_0G(x)$.

Now let $\xi \in CC_0G(x)$. This implies in the sequel $0 \in C_0G(x - \dot{\xi})$ and

$$0 \in G(x - \dot{\xi}). \tag{3.3}$$

Since $\xi \in G(\dot{\xi})$, in view of S, we obtain from (3.3) the relation

$$\xi = 0 + \xi \in G(x - \dot{\xi}) + G(\dot{\xi}) \subset G(x),$$

by virtue of A. Thus identity (3.1) is proved.

Note that identity (3.2) requires assuming axioms S and A for G, but the inclusion

$$G \subset C\bar{C}_0G \tag{3.4}$$

holds generally.

PROPOSITION 3.4. *Relation* (3.4) *holds for every general convergence.*

P r o o f. Let $\xi \in G(x)$. Of course,

$$0 = \xi - \xi \in G(x - \dot{\xi} + \dot{\xi}) - \xi$$

and, by the definition of $\bar{C}_0G$, we have

$$0 \in \bar{C}_0G(x - \dot{\xi}),$$

i.e., $\xi \in C\bar{C}_0G(x)$. The proof is finished.

Now we shall show that identity (3.1) and the inclusion

$$C\bar{C}_0G \subset G$$

are false, if one axioms S, A is not satisfied by G.

EXAMPLE 3.1. Let G be as in Example 1.1. As we have noticed, G satisfies A, but not S. We have $C_0G(x) = \{0\}$ if $x = \{\xi_n\}$, $\xi_n \to 0$ and $C_0G(x) = \emptyset$ for other sequences x. Further, $CC_0G(x) = \{\xi\}$ if $x = \{\xi_n\}$, $\xi_n \to \xi$ and $CC_0G(x) = \emptyset$ otherwise. Therefore

$$\{-\xi\} = G(x) \subsetneq CC_0G(x) = \{\xi\},$$
$$\{\xi\} = CC_0G(x) \subsetneq G(x) = \{-\xi\}$$

for $x = \{\xi_n\}$, $\xi_n \to \xi \neq 0$.

Now, it is easy to see that $\bar{C}_0G(x) = \{0\}$ if $\xi_n \to \xi$ for some $\xi \in R$ and $\bar{C}_0G(x) = \emptyset$ otherwise. Hence $C\bar{C}_0G(x) = R$ if $\xi_n \to \xi$ ($\xi \in R$), i.e., for such a sequence $x = \{\xi_n\}$ we have

$$C\bar{C}_0G(x) \subsetneq G(x).$$

EXAMPLE 3.2. Let G be as in Example 1.2. As we have seen, G satisfies S and not A. We have

$$CC_0G(x) = \begin{cases} \{\xi\} & \text{if } \xi_n = \xi \text{ for almost all } n \\ \emptyset & \text{otherwise} \end{cases}$$

and thus

$$\{\xi\} = G(x) \subsetneq CC_0G(x) = \emptyset$$

if $\xi_n = 1 + \frac{1}{n}$, for instance.

EXAMPLE 3.3. Let $E = R$. We define a general convergence G as follows: if $\xi_n \to 0$, then we adopt $G(x) = \{0\}$; if $\xi_n = \xi$ for almost all n, then $G(x) = \{\xi\}$; in the remaining cases $G(x) = \emptyset$. Obviously, G satisfies S and does not satisfy A.

Let $x = \{\xi_n\}$, where $\xi_n = \xi + \frac{1}{n}$ with $\xi \in R$, $n \in N$. Then we have $CC_0G(x) = \{\xi\} = C\bar{C}_0G(x)$, that is,

$$CC_0G(x) \subsetneq G(x)$$

and

$$C\bar{C}_0G(x) \subsetneq G(x).$$

PROPOSITION 3.5. *If a zero-convergence* G_0 *on* E *satisfies axiom* T, *the*

(3.5) $$G_0 \subset C_0CG_0$$

and

(3.6) $$G_0 \subset \bar{C}_0CG_0.$$

P r o o f. To prove (3.5), suppose that $\xi \in G_0(x)$. This means that

(3.7) $$0 \in G_0(x)$$

and, by T, that

(3.8) $$0 \in G_0(x - \dot{\xi}).$$

By the definition of C, we get from (3.7) and (3.8)

$$\xi, 0 \in CG_0(x).$$

Hence

(3.9) $$\xi \in C_0CG_0(x) = CG_0(x)$$

and

(3.10) $$\eta, \xi+\eta \in CG_0(x+\dot{\eta})$$

for every $\eta \in E$.

From (3.10), we obtain

(3.11) $$\xi \in \bar{C}_0CG_0(x).$$

Relations (3.9) and (3.11) prove our assertion.

Since for zero-convergences

(3.12) H implies T,

we have immediately

COROLLARY 3.1. *If a zero-convergence G_0 on E satisfies axiom* H, *then relations* (3.5) *and* (3.6) *hold.*

Now, we shall show that relations (3.5) and (3.6) are not true generally (if G_0 does not satisfy H or T).

EXAMPLE 3.4. Let E be an arbitrary linear space and let $G_0(x) = E$ if $\xi_n = 0$ for almost all n. For remaining sequences x, we put $G_0(x) = \emptyset$. Note that H does not hold.

We have

$$CG_0(x) = \begin{cases} \{\xi\} & \text{if } \xi_n = \xi \text{ for almost all } n \\ \emptyset & \text{otherwise} \end{cases}$$

and thus

$$C_0CG_0(x) = \bar{C}_0CG_0(x) = \{0\}$$

if $\xi_n = \xi$ for almost all n. This means that (3.5) and (3.6) are false in this case.

PROPOSITION 3.6. *If a zero-convergence G_0 satisfies* T′, *then*

(3.13) $$C_0CG_0 \subset G_0$$

and

(3.14) $$\bar{C}_0CG_0 \subset G_0.$$

P r o o f. If $\xi \in C_0CG_0(x)$, then in turn

$$\xi \in CG(x)$$

and

$$0 \in G_0(x-\dot{\xi}).$$

The last relation, by T′, implies that $\xi \in G_0(x)$ and (3.13) holds.

If $\xi \in C_0CG_0(x)$, then there exists an $\eta \in E$ such that $\eta, \xi+\eta \in CG_0(x+$ $+\dot{\eta})$ and hence $0 \in G_0(x+\dot{\eta})$. This implies $\xi \in G_0(x)$, by T′ and (3.14) is proved.

Since axiom T′ is somewhat artificial for zero-convergences (see section 4), we shall prove (3.13) and (3.14) also under other assumptions.

PROPOSITION 3.7. *If a convergence G_0 satisfies* S_0', A *and* M, *then* (3.13) *and* (3.14) *hold.*

P r o o f. By Proposition 2.3, the general convergence CG_0 and the zero-convergence C_0CG satisfy axiom H. To prove (3.13), it remains to note that $0 \in C_0CG_0(x)$ implies $0 \in CG_0(x)$ and this implies $0 \in G_0(x)$.

Relation (3.14) is obvious if $G_0 = \emptyset$. Further, note that for non-empty zero-convergences condition M implies S_0. Hence, by Propositions 2.4 and 2.6, the general convergence CG_0 fulfils axioms S and A. Consequently, we have $C_0CG_0 = C_0CG_0$, in view of the second part of Proposition 1.1. Hence (3.14) follows, by virtue of the first part of this proposition.

Note that for zero-convergences

$$\text{(3.15)} \qquad \text{T}' \text{ implies T.}$$

In fact, assume that $\xi \in G_0(x)$. Then $0 \in G_0(x) = G_0(x-\dot{\xi}+\dot{\xi})$ and we obtain

$$-\xi \in G_0(x-\dot{\xi}),$$

in view of T′. But this means, by the definition of zero-convergences, that $0 \in G_0(x-\dot{\xi})$. Thus (3.15) holds.

By virtue of implications (3.12) and (3.15), we obtain from Propositions 3.5, 3.6 and 3.7 the following result.

COROLLARY 3.2. *If a zero-convergence G_0 on E satisfies* 1° T′; *or* 2° S′, A, M *and* T; *or* 3° S_0', H, A *and* M, *then the identities*

$$\text{(3.16)} \qquad C_0CG_0 = G_0$$

and

$$\text{(3.17)} \qquad \bar{C}_0CG_0 = G_0$$

hold.

We shall show now that relations (3.13)—(3.14) and (3.16)—(3.17) are not true generally (if T′ and one of axioms S_0', A, M do not hold for G_0).

EXAMPLE 3.5. If G_0 is as in Example 2.1, then axioms H, A, M are satisfied and axioms S_0', T′ — not. We have

$$C_0CG_0(\dot{\xi}) = \bar{C}_0CG_0(\dot{\xi}) = R \subsetneq \{0\} = G_0(\dot{\xi})$$

for every $\xi \in X$.

If G_0 is taken as in Example 2.2, then H, M, S_0' hold and, at the same time, A as well as T′ do not hold. Moreover we have

$$C_0CG_0\left(\left\{\xi+\frac{1}{n}\right\}\right) = \bar{C}_0CG_0\left(\left\{\xi+\frac{1}{n}\right\}\right) = R \supsetneq \{0\} = G_0\left(\left\{\xi+\frac{1}{n}\right\}\right)$$

for every $\xi \in X$.

At last, if G_0 is taken from Example 2.3, then H, A, S_0' are fulfilled, but any of axioms M, T′ is not. We have in this case

$$C_0CG_0(x) = \bar{C}_0CG_0(x) = R \supsetneq \{0\} = G_0(x)$$

for $x = \{\xi_n\}$ with $\xi_n \to \infty$.

4. Finally, we would like to present, without proofs, mutual relations between axioms concerning general convergences.

First note that each of axioms F, U, S, H, A, M is independent of others. Axioms F, U, A, M do not depend on axioms S′, T, T′ either. However we have the following implication:

(4.1) $$S' \wedge A \wedge M \Rightarrow H.$$

On the other hand, according to (4.1) it can be shown that H does not depend: 1° on F, U, S, A, M, T and T′; 2° on F, U, S, S′, M, T and T′.

Further note that S does not depend on axioms F, U, H, A, M, S′, T and T′, because the trivial convergence (defined as $G(x) = \emptyset$ for all $x \in X^N$) satisfies all mentioned axioms except S. The situation is different when considering only non-trivial convergences. Then we have the implication

(4.2) $$M \wedge T' \Rightarrow S.$$

On the other hand, one can show that in the class of nontrivial convergences S does not depend: 1° on F, U, S′, H, A, M and T; 2° on F, U, S′, H, A, T and T′.

We are passing now to axioms S′, T, T′. The following relations hold:

(4.3) $$S \wedge H \Rightarrow S',$$

(4.4) $$H \wedge M \wedge T' \Rightarrow S',$$

(4.5) $$H \wedge A \wedge T \wedge T' \Rightarrow S',$$

(4.6) $$S \wedge A \Rightarrow T,$$

(4.7) $$S \wedge A \Rightarrow T'.$$

However

1° axiom S′ does not depend: a) on F, U, S, A, M, T, T′; b) on F, U, H, A, M, T; c) on F, U, H, A, T; d) on F, U, H, T, T′ — see (4.3), (4.4), (4.5) and (4.2);

2° axiom T does not depend: a) on F, U, S, S′, H, M, T′; b) on F, U, S′, H, A, T′; c) on F, U, S′, H, A, M — see (4.6) and (4.2);

3° axiom T′ does not depend: a) on F, U, S, S′, H, M, T; b) on F, U, S′, H, A, M, T — see (4.7).

Finally note that axiom T′ is unnatural for zero-convergences. For we have the implication

$$S_0 \wedge S'_0 \Rightarrow \sim T',$$

provided $E \neq \{0\}$.

We omit here other relations between axioms S_0, S'_0 and the remaining ones for zero-convergences.

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