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Report of Meeting

The Twenty-second Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities Hajdúszoboszló (Hungary), February 1–4, 2023

The Twenty-second Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities was held in Hotel Aurum, Hajdúszoboszló, Hungary, from February 1 to February 4, 2023. It was organized by the Department of Analysis of the Institute of Mathematics of the University of Debrecen.

The Winter Seminar was supported by the Institute of Mathematics, University of Debrecen and by the project 2019-2.1.11-TÁT-2019-00049.

The 27 participants came from the University of Silesia (Poland), the University of Debrecen (Hungary), the University of Rzeszów (Poland) and the Kazimierz Wielki University (Poland), 8 from the first, 16 from the second, 2 from the third and 1 from the fourth university.

Professor Zsolt Páles opened the Seminar and welcomed the participants to Hajdúszoboszló.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iterative equations, equations on algebraic structures, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity and Walsh–Fourier analysis. Interesting discussions were generated by the talks.

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The closing address was given by Professor Maciej Sablik. His invitation to the Twenty-third Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities in January 2024 in Poland was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in Section 1, problems and remarks in chronological order in Section 2, and the list of participants in the final section.

1. Abstracts

Ali Hasan Ali: *Generalizations of the Taylor Theorem with Factorization Results* (Joint work with Zsolt Páles)

In this work, we present an extension of the Taylor theorem for linear differential operators with constant coefficients. Our approach is based on the use of divided differences with repeated arguments. The extension of the Taylor theorem for exponential polynomials is established, along with its consequences in terms of integral remainder terms and mean value type theorems. As an application, we provide some factorization results by obtaining estimates for linear functionals through the use of the generalized Taylor theorem.

Reference

[1] A.H. Ali and Zs. Páles, *Taylor-type expansions in terms of exponential polynomials*, Math. Inequal. Appl. **25** (2022), no. 4, 1123–1141.

Mihály Bessenyei: *Fixed point aspects of the Presić equation* (Joint work with Zsolt Páles)

Let H and X be nonempty sets, further, $F\colon H\times X^n\to X$ and $\varphi_1,\dots,\varphi_n : H\to H$ be given functions. In the talk, we present existence and uniqueness results to the solvability of the Presić-type equation

$$f(t)=F\bigl(t,f(\varphi_1(t)),\dots,f(\varphi_n(t))\bigr).$$

The approach is based on fixed point methods. The Bielecki renorming technique, the Banach contraction principle and the Tarski fixed point theorem play the key role in the proofs.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.
[2] M. Bessenyei, G. Horváth, and Cs.G. Kézi, *Functional equations on finite groups of substitutions*, Expo. Math. **30** (2012), no. 3, 283–294.
[3] M. Bessenyei, Á. Konkoly, and G. Szabó, *Linear functional equations involving finite substitutions*, Acta Sci. Math. (Szeged) **83** (2017), no. 1–2, 71–81.
[4] M. Bessenyei and Zs. Páles, *Applications of the Bielecki renorming technique*, J. Fixed Point Theory Appl. **23** (2021), no. 2, Paper No. 15, 23 pp.
[5] A. Bielecki, *Une remarque sur la méthode de Banach–Cacciopoli–Tikhonov dans la théorie des équations différentielles ordinaires*, Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 261–264.
[6] S.B. Prešić, *Méthode de résolution d'une classe d'équations fonctionnelles linéaires*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. **No. 119** (1963), 21–28.
[7] S.B. Prešić, *Sur l'équation fonctionnelle* $f(x) = H(x, f(x), f(\theta_2 x), \dots, f(\theta_n x))$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. **No. 118** (1963), 17–20.
[8] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math. **5** (1955), 285–309.

Zoltán Boros: *Applications of strong geometric derivatives* (Joint work with Péter Tóth)

Let I denote an open interval in the real line, $0 \le \varepsilon \in \mathbb{R}$ and $1 < p \in \mathbb{R}$. Let us consider a function $f\colon I \to \mathbb{R}$ that fulfills the inequality

$$(1)\quad f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \le f(x) + f(y) + \varepsilon\,(\lambda(1-\lambda)|x-y|)^p$$

for every $x, y \in I$ and $\lambda \in [0,1]$. We prove that such a function f has increasing strong geometric derivatives (cf. the presentation by P. Tóth [1]). According to the representation theorem for this property [1, Theorem 2], there exist a convex function $g\colon I \to \mathbb{R}$ and an additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + A(x)$ for every $x \in I$. This implies, in particular, that f is Wright-convex [4], i.e., it satisfies inequality (1) with $\varepsilon = 0$ as well.

Such an argument provides a new proof for Ng's decomposition theorem [2] (in the particular case when the domain is a real interval), as well as a Rolewicz type result [3] and a localization principle for Wright-convexity.

Applying a decomposition theorem for strongly geometrically differentiable functions, we perform similar investigations concerning locally approximately affine mappings.

References

[1] Z. Boros and P. Tóth, *Strong geometric derivatives*, Abstract of presentation by P. Tóth during the 22^{nd} DKWS (the 22^{nd} Debrecen–Katowice Winter Seminar, Hajdúszoboszló, Hungary, February 1–4, 2023).

[2] C.T. Ng, *Functions generating Schur-convex sums*, in: W. Walter (ed.), *General Inequalities 5: 5th International Conference on General Inequalities, Oberwolfach, May 4–10, 1986*, Internat. Schriftenreihe Numer. Math. **80**, Birkhäuser, Basel, 1987, pp. 433–438.
[3] S. Rolewicz, *On γ-paraconvex multifunctions*, Math. Japon. **24** (1979), 293–300.
[4] E.M. Wright, *An inequality for convex functions*, Amer. Math. Monthly **61** (1954), 620–622.

Jacek Chudziak: *Solutions of a composite type functional equation on a cone* (Joint work with Zdenék Kočan)

Let M be a non-empty set equipped with a binary operation $\circ\colon M \times M \to M$. In a recent paper [1], the solutions of the functional equation

$$f(x+y) = f(x) \circ f(h(x)y) \quad \text{for} \quad x, y \in [0,\infty), \tag{1}$$

where $f\colon [0,\infty) \to M$ and $h\colon [0,\infty) \to [0,\infty)$ are unknown functions, have been investigated in a connection with some invariance problems under binomial thinning. In the talk, we are going to deal with the following generalization of equation (1)

$$f(x+y) = f(x) \circ f(h(x)y) \quad \text{for} \quad x, y \in \mathcal{C}, \tag{2}$$

where $\mathcal{C}$ is a convex cone in a real linear space, $f\colon \mathcal{C} \to M$ and $h\colon \mathcal{C} \to [0,\infty)$.

Reference

[1] K. Baron and J. Wesołowski, *From invariance under binomial thinning to unification of the Cauchy and the Gołąb–Schinzel-type equations*, Results Math. **76** (2021), Paper No. 168, 14 pp.

György Gát: *Means of subsequences of partial sums of trigonometric Fourier-series*

In this talk, we give a short résumé with respect to some recent results in the theory of summation of trigonometric Fourier-series. We discuss the case when only a subsequence of the sequence of partial sums is given ([2], [3]). We examine for which index sequences and in what sense the original function can be reconstructed. For convex index sequences and continuous functions (supremum norm) Carleson [1] has given a necessary and sufficient condition. His result is partly based on two theorems of Kahane and Katznelson [4]. For integrable functions and almost everywhere convergence, the situation is considerably more difficult. Finally, we formulate some problems and conjectures in this research area of Fourier analysis.

References

[1] L. Carleson, *Appendix to the paper by J.-P. Kahane and Y. Katznelson*, in: P. Erdős et al. (eds.), *Studies in Pure Mathematics*, Birkhäuser, Basel-Boston, 1983, pp. 411–413.
[2] G. Gát, *Cesàro means of subsequences of partial sums of trigonometric Fourier series*, Constr. Approx. **49** (2019), no. 1, 59–101.
[3] G. Gát, *Almost everywhere divergence of Cesàro means of subsequences of partial sums of trigonometric Fourier series*, submitted.
[4] J.-P. Kahane and Y. Katznelson, *Séries de Fourier des fonctions bornées*, in: P. Erdős et al. (eds.), *Studies in Pure Mathematics*, Birkhäuser, Basel-Boston, 1983, pp. 395–410.

Attila Gilányi: *Computer aided investigation of alienness of linear functional equations* (Joint work with Lan Nhi To)

The concepts of the alienness and the strong alienness of functional equations were introduced by Jean Dhombres in his paper [3]. These properties were studied by several authors during the last more than 30 years (cf., e.g., [4] and [9]).

In this talk, we consider linear functional equations of the form

$$\sum_{i=0}^{n+1} f_i(p_i x + q_i y) = 0 \qquad (x, y \in X), \tag{1}$$

where n is a positive integer, $p_0, \dots, p_{n+1}$ and $q_0, \dots, q_{n+1}$ are rational numbers, X, Y are linear spaces and $f_0, \dots, f_{n+1} \colon X \to Y$ are unknown functions. Extending the results presented in [6], we describe a computer program package developed in the computer algebra system Maple (Maple is a trademark of Waterloo Maple Inc.), which is able to make a decision about alienness or strong alienness of functional equations belonging to class (1). The package is based on another Maple program, which determines the solutions of linear functional equations of type (1) (cf., [2], furthermore, [1], [5], [7] and [8]).

References

[1] G.Gy. Borus and A. Gilányi, *Solving systems of linear functional equations with computer*, 4^{th} IEEE International Conference on Cognitive Infocommunications (CogInfoCom), IEEE, 2013, 559–562.
[2] G.Gy. Borus and A. Gilányi, *Computer assisted solution of systems of two variable linear functional equations*, Aequationes Math. **94** (2020), no. 4, 723–736.
[3] J. Dhombres, *Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Math. **35** (1988), no. 2-3, 186–212.
[4] R. Ger and M. Sablik, *Alien functional equations: a selective survey of results*, in: J. Brzdęk et al. (eds.), *Developments in Functional Equations and Related Topics*, Springer Optim. Appl. **124**, Springer, Cham, 2017, pp. 107–147.
[5] A. Gilányi, *Charakterisierung von monomialen Funktionen und Lösung von Funktionalgleichungen mit Computern*, Diss., Universität Karlsruhe, Karlsruhe, Germany, 1995.

[6] A. Gilányi, *Alienness of linear functional equations, The Nineteenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Zakopane (Poland), January 30 – February 2, 2019*, Ann. Math. Sil. **33** (2019), p. 311.
[7] L. Székelyhidi, *On a class of linear functional equations*, Publ. Math. Debrecen **29** (1982), no. 1-2, 19–28.
[8] L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.
[9] T. Szostok, *Alienation of two general linear functional equations*, Aequationes Math. **94** (2020), no. 2, 287–301.

Richárd Grünwald: *Local and global inequalities for nonsymmetric generalized Bajraktarević means* (Joint work with Zsolt Páles)

In this talk, the ith entry of a real vector

$$x := (x_i)_{i\in\{1,\dots,n\}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

will be denoted by x_i, and analogously, the ith row and jth column of a real matrix

$$x := (x_i^j)_{(i,j)\in\{1,\dots,n\}\times\{1,\dots,k\}} = \begin{pmatrix} x_1^1 & \cdots & x_1^k \\ \vdots & & \vdots \\ x_n^1 & \cdots & x_n^k \end{pmatrix} \in \mathbb{R}^{n\times k}$$

will be denoted by x_i and x^j, respectively. For convenience, we identify $\mathbb{R}^{n\times k}$ by $(\mathbb{R}^n)^k$ in the standard manner.

Firstly we give 2 necessary conditions for the local validity of the functional inequality

$$M_0(\Phi(x_1),\dots,\Phi(x_n)) \le \Phi(M_1(x^1),\dots,M_k(x^k)), \tag{1}$$

where, for $\alpha \in \{0,\dots,k\}, I_\alpha \subseteq \mathbb{R}$ is a nonempty open interval, $M_\alpha\colon I_\alpha^n \to I_\alpha$ is a mean and $\Phi\colon I \to I_0$, where $I := I_1 \times \cdots \times I_k$, assuming differentiability and twice differentiability, respectively.

Then we show necessary and sufficient conditions for the particular case of (1) when all the means are n-variable nonsymmetric generalized Bajraktarević means, i.e., we consider the inequality

$$A_{f_0,p^0}(\Phi(x_1),\dots,\Phi(x_n)) \le \Phi(A_{f_1,p^1}(x^1),\dots,A_{f_k,p^k}(x^k)).$$

References

[1] M. Bajraktarević, *Sur une équation fonctionnelle aux valeurs moyennes*, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **13** (1958), 243–248.
[2] M. Bajraktarević, *Sur une généralisation des moyennes quasilinéaires*, Publ. Inst. Math. (Beograd) (N.S.) **3(17)** (1963), 69–76.
[3] R. Grünwald and Zs. Páles, *On the equality problem of generalized Bajraktarević means*, Aequationes Math. **94** (2020), no. 4, 651–677.
[4] R. Grünwald and Zs. Páles, *Local and global inequalities for nonsymmetric generalized Bajraktarević means*, manuscript.
[5] Zs. Páles and A. Zakaria, *On the local and global comparison of generalized Bajraktarević means*, J. Math. Anal. Appl. **455** (2017), no. 1, 792–815.

Eszter Gselmann: *On Iverson's law of similarity* (Joint work with Christopher W. Doble and Yung-Fong Hsu)

Consider an experimental context in which a participant must compare a stimulus of intensity x with one of intensity y (both measured in ratio scale units) and judge which has the greater sensory impact (i.e., is louder, is heavier, is brighter, etc.). Let $P(x,y)$ be the probability that intensity y is judged greater than intensity x. The problem of Fechner [1], stated in modern terms, is to find continuous and strictly increasing functions u and F such that

$$P(x,y) = F\left(u(x) - u(y)\right).$$

The interpretation of this equation is that the stimulus intensities x and y are scaled by the sensory mechanism to the values $u(x)$ and $u(y)$, respectively, and the probabilities $P(x,y)$ are determined by the differences between those scaled values. The function u is called a *scale*, and the above equation is a *Fechnerian representation.*

Fixing the probability π, one can define the function σ_π by

$$\sigma_\pi(x) = y \quad \text{if and only if} \quad P(x,y) = \pi.$$

That is, $\sigma_\pi(x)$ is the intensity in the second interval judged greater than intensity x in the first interval with probability π. We employ $F^{-1}(\pi) = s$ as the index measuring discriminability between stimuli. Following Iverson [2], we define

$$\xi_s(x) = \sigma_\pi(x)$$

and call the functions ξ_s *sensitivity functions.* In this talk, sensitivity functions will be considered that also fulfill *Iverson's law of similarity*, i.e.,

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x).$$

REFERENCES

[1] G. Fechner, *Elements of Psychophysics*, Holt, Rinehart & Winston, New York, 1966. (Original work published 1860.)
[2] G.J. Iverson, *Analytical methods in the theory of psychophysical discrimination. II: the near-miss to Weber's law, Falmagne's law, the psychophysical power law and the law of similarity*, J. Math. Psych. **50** (2006), no. 3, 283–289.

MEHAK IQBAL: *Monomial functions, normal polynomials and polynomial equations* (Joint work with Eszter Gselmann)

Let $\mathbb{F}$ be a subfield of the complex field. In this talk, we consider generalized monomial functions $f, g\colon \mathbb{F} \to \mathbb{C}$ (of possibly different degree) that also fulfill

$$f(P(x)) = Q(g(x)) \quad (x \in \mathbb{F}),$$

where $P \in \mathbb{F}[x]$ and $Q \in \mathbb{C}[x]$ are given (classical) polynomials.

In more detail, during the talk, we focus on the following.

(i) We study generalized monomials $f\colon \mathbb{F} \to \mathbb{C}$ of degree n under the condition that the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a (normal) polynomial.

(ii) As an inital step we show that instead of polynomials P, we always may restrict ourselves to (classical) monomials.

(iii) At first glance the assumption that the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a (normal) polynomial, seems a bit artificial. Nevertheless, as some illustrative examples show, this is not the case. They also reveal why homomorphisms and higher order derivations appear in such characterization theorems.

TIBOR KISS: *A functional equation with arithmetic mean*

The talk discusses the regular and irregular solutions of the Pexider functional equation

$$\varphi\Big(\frac{x+y}{2}\Big)\big(f_1(x) - f_2(y)\big) = 0, \quad (x, y) \in I \times J,$$

where $I, J \subseteq \mathbb{R}$ stand for nonempty open intervals and $\varphi\colon \frac{1}{2}(I + J) \to \mathbb{R}$, $f_1\colon I \to \mathbb{R}$, and $f_2\colon J \to \mathbb{R}$ are assumed to be unknown functions. We also touch on why the above equation plays an essential role in the problem of the invariance of generalized weighted quasi-arithmetic means.

References

[1] Z. Daróczy and M. Laczkovich, *On functions taking the same value on many pairs of points*, Real Anal. Exchange **33** (2007/2008), no. 2, 385–394.
[2] Z. Daróczy and Zs. Páles, *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen **61** (2002), no. 1–2, 157–218.
[3] Z. Daróczy and V. Totik, *Remarks on a functional equation*, Acta Sci. Math. (Szeged) **81** (2015), no. 3–4, 527–534.
[4] T. Kiss and Zs. Páles, *On a functional equation related to two-variable Cauchy means*, Math. Inequal. Appl. **22** (2019), no. 4, 1099–1122.

Radosław Łukasik: *On some functional equation in a single variable*

Marian Tetiva [1] has posed the following problem: Let a and b be real numbers with $0 < a < 1 < b$. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and

$$f(f(x)) - (a+b)f(x) + abx = 0 \tag{1}$$

for all $x \in \mathbb{R}$. We present the general solution of (1) in the class of functions with the Darboux property from the interval I onto I $(0 \in I)$, where $a, b \in \mathbb{R}$.

Reference

[1] *Problem 12347*, Amer. Math. Monthly **129** (2022), no. 8, p. 786.

Oleksandr Maslyuchenko: *Compact subspaces of the space of separately continuous functions with the cross-uniform topology and the sharp cellularity* (Joint work with Roman Ivasiuk)

A natural topology on the space S of all separately continuous functions $f\colon [0,1]^2 \to \mathbb{R}$ was introduced in [1] and was called *the topology of the sectionally uniform convergence*. This topology can be considered on the space $S(X \times Y)$ of all separately continuous functions $f\colon X \times Y \to \mathbb{R}$ for any topological spaces X and Y. Its base consists of the sets

$$W_{E,\varepsilon}(f_0) = \Big\{ f \in S(X \times Y) : |f(p) - f_0(p)| < \varepsilon \text{ for any } p \in \mathrm{cr}E \Big\},$$

where E is a finite subset of $X \times Y$, $\mathrm{cr}E = (X \times \mathrm{pr}_Y(E)) \cup (\mathrm{pr}_X(E) \times Y)$ is *the cross* of the set E, $\varepsilon > 0$ and $f_0 \in S(X \times Y)$. We call it *the cross-uniform topology* and always endow the space $S(X \times Y)$ by it. If X and Y are compacta then $S(X\times Y)$ is a topological vector space. In [1], it was proved only that $S = S([0,1]^2)$ is a separable non-metrizable complete topological vector space, and the authors asked about the other properties of S. In [2, 3, 4], the authors proved that $S(X\times Y)$ is a meager, complete, barreled and bornological topological vector space for any compacta X and Y without isolated points.

Another intrigued question is the problem on description of compact subspaces of $S(X\times Y)$ for any compacta X and Y. Compact subspaces of $B_1(X)$ (= the space of all Baire one function with the pointwise topology) are, so-called, Rosenthal compacta if X is a Polish space. Since $S([0,1]^2) \subseteq B_1([0,1]^2)$ and every Baire one function on the diagonal can be extended to the separately continuous function on the whole square, we expected the appearance of some Rosenthal type compacta. But it turns out that the structure of compact subspaces of $S(X \times Y)$ is simpler. Let $w(X)$ denote the weight of a topological space X and let $c(X)$ denote the cellularity of X. *The sharp cellularity* is $c^\sharp(X) = \sup\Big\{|\mathcal{U}|^+ : \mathcal{U} \text{ is a disjoint family of open sets in } X\Big\}$, where $|A|$ means the cardinality of a set A and $\mathfrak{m}^+$ means the least cardinal number which is grater then the cardinal number $\mathfrak{m}$. The main result of our talk is the following.

Theorem 1. *Let X, Y be infinity compacta and K be a compact. Then K embeds into $S(X \times Y)$ if and only if $w(K) < \min\{c^\sharp(X), c^\sharp(Y)\}$.*

References

[1] H.A. Voloshyn and V.K. Maslyuchenko, *The topologization of the space of separately continuous functions*, Carpathian Math. Publ. **5** (2013), no. 2, 199–207.

[2] H.A. Voloshyn, V.K. Maslyuchenko, and O.V. Maslyuchenko, *Embedding of the space of separately continuous functions into the product of Banach spaces and its barrelledness*, Mathematical Bulletin of Shevchenko Scientific Society **11** (2014), 36–50.

[3] H.A. Voloshyn, V.K. Maslyuchenko, and O.V. Maslyuchenko, *The bornologicity of the space of separately continuous functions*, Mathematical Bulletin of Taras Shevchenko Scientific Society **12** (2015), 61–67.

[4] H.A. Voloshyn, V.K. Maslyuchenko, and O.V. Maslyuchenko, *On Baireness of the space of separately continuous function*, Transactions of Institute of Mathematics, the NAS of Ukraine **12** (2015), no. 3, 78–96.

Rayene Menzer: *An alternative equation for generalized monomials involving measure* (Joint work with Zoltán Boros)

Motivated by analogous investigations for additive functions [1], in this talk we consider a generalized monomial $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies the additional equation $f(x)f(y) = 0$ for the pairs $(x, y) \in D$, where $D \subset \mathbb{R}^2$ has positive planar Lebesgue measure. We prove that $f(x) = 0$ for all $x \in \mathbb{R}$. Using analogous arguments, we establish a related statement about the signs of such functions: if a generalized monomial f of an even degree is non-negative on a measurable subset of reals with positive Lebesgue measure, then $f(x) \geq 0$ for every real number x.

Reference

[1] Z. Kominek, L. Reich, and J. Schwaiger, *On additive functions fulfilling some additional condition*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **207** (1998), 35–42 (1999).

Gábor Marcell Molnár: *On polyhedral maps and some of their applications* (Joint work with Zsolt Páles)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces over $\mathbb{R}$. We say that $T\colon X \to Y$ is a *polyhedral map* if T is continuous and there exist bounded linear maps $A_1, \dots, A_n\colon X \to Y$ and $y_1, \dots, y_n \in Y$ such that

$$T(x) \in \{A_1(x) + y_1, \dots, A_n(x) + y_n\} \quad (x \in X).$$

In the talk, I will show some results regarding such polyhedral maps and an application in finite dimensional setting.

Reference

[1] G.M. Molnár and Zs. Páles, *On convex and concave sequences and their applications*, Math. Inequal. Appl. **25** (2022), no. 3, 727–750.

Andrzej Olbryś: *On a joint generalization of P-functions and quasi-convex functions*

Let D be a convex subset of a real linear space, $I \subset \mathbb{R}$. In our talk, we study the properties of functions $f : D \to I$ satisfying the functional inequality

$$f(tx + (1-t)y) \leq T(f(x), f(y)), \quad x, y \in D,\ t \in [0,1],$$

where a binary operation $T\colon I \times I \to I$ is associative, symmetric and increasing with respect to the first (or second) variable.

References

[1] K. Baron, J. Matkowski, and K. Nikodem, *A sandwich with convexity*, Math. Pannon. **5** (1994), no. 1, 139–144.
[2] W. Förg-Rob, K. Nikodem, and Zs. Páles, *Separation by monotonic functions*, Math. Pannon. **7** (1996), no. 2, 191–196.
[3] K. Nikodem, *Approximately quasiconvex functions*, C. R. Math. Rep. Acad. Sci. Canada **10** (1988), no. 6, 291–294.
[4] C.E.M. Pearce and A.M. Rubinov, *P-functions, quasi-convex functions, and Hadamard-type inequalities*, J. Math. Anal. Appl. **240** (1999), no. 1, 92–104.

Zsolt Páles: *On the Jensen convexity of quasideviation and Bajraktarević means* (Joint work with Paweł Pasteczka)

The Jensen convexity of quasiarithmetic means has been completely characterized in a recent paper [1] in the following manner: A quasiarithmetic mean over an open interval I generated by a strictly monotone continuous function $f\colon I \to \mathbb{R}$ is Jensen convex if and only if f is twice continuously differentiable with a nowhere vanishing first derivative and either f'' is nonvanishing and f'/f'' is positive and concave, or f'' is identically zero. Motivated by this result, we characterize the Jensen convexity of quasideviation and Bajraktarević means in a similar spirit.

Reference

[1] Zs. Páles and P. Pasteczka, *On the Jensen convex and Jensen concave envelopes of means*, Arch. Math. (Basel) **116** (2021), no. 4, 423–432.

Evelin Pénzes: *Dimension of generalized fractals in semimetric spaces* (Joint work with Mihály Bessenyei)

In this talk we will present a correspondence of Hutchinson's Fractal Dimension Theorem: In the well known exponential expression for the Hausdorff dimension, the factors of contractions are replaced by the appropriate Dini derivatives of the generalized contractions.

Reference

[1] M. Bessenyei and E. Pénzes, *Generalized fractals in semimetric spaces*, Nonlinear Anal. **220** (2022), Paper No. 112853, 15 pp.

Mateusz Pieszczek: *Sincov's Equation for Fuzzy Implications with Fuzzy Connectives* (Joint work with Michał Baczyński)

The Sincov's equation was described at the beginning of 20th century in [5], but at the time it was not used in context of fuzzy connectives. In article [4], the authors discovered that characterization of T-power based implications is naturally using Sincov's functional equation. As Gronau describes in [3], classical multiplicative Sincov's equation is given by

$$I(x,y) \cdot I(y,z) = I(x,z),$$

where, in our context, I is a fuzzy implication function. Importance of T-power based implications, cannot be understated as they are invariant with respect to powers of T-norms. This property is sometimes a required assumption when designing fuzzy inference systems, which uses implication that preserves "gradation". For instance, if an implication used to model rules below was invariant with respect to T-powers, the stated rules would be equally true.

If the apple is **red**, then it is **ripe**.

If the apple is *very* **red**, then it is *very* **ripe**.

If the apple is *little* **red**, then it is *little* **ripe**.

Later in [1], the authors provided full characterization of solutions to multiplicative Sincov's equation for fuzzy implications. Then in [2], they extended the result to a characterization of the equation with T-norms. In this talk, we consider generalized Sincov's equation for fuzzy implications using class of generalized fuzzy connectives instead of multiplication or T-norms. Our main result considers a characterization of solutions for representable uninorms, where the equation takes the form of

$$U(I(x,y), I(y,z)) = I(x,z).$$

References

[1] M. Baczyński, W. Fechner, and S. Massanet, *A functional equation stemming from a characterization of power-based implications*, in: *2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, IEEE, 2019, pp. 1–6.
[2] M. Baczyński, W. Fechner, and S. Massanet, *On a generalization of multiplicative Sincov's equation for fuzzy implication functions*, Fuzzy Sets and Systems **451** (2022), 196–205.
[3] D. Gronau, *A remark on Sincov's functional equation*, Notices of the South African Mathematical Society **31** (2000), no.1, 1–8.
[4] S. Massanet, J. Recasens, and J. Torrens, *Some characterizations of T-power based implications*, Fuzzy Sets and Systems **359** (2019), 42–62.
[5] D.M. Sincov, *Bemerkungen über Funktionalrechnung*, (Russian), Bull. Soc. Phys.-Math. Kazan (2) **13** (1903), 48–72.

Maciej Sablik: *Local polynomials* (Joint work with Chisom P. Okeke)

We deal with a characterization of polynomial functions defined in convex subsets of linear spaces by functional equations. We illustrate our method by giving some examples.

References

[1] Z. Daróczy and Gy. Maksa, *Functional equations on convex sets*, Acta Math. Hungar. **68** (1995), no. 3, 187–195.
[2] J. Ger, *On Sahoo-Riedel equations on a real interval*, Aequationes Math. **63** (2002), no. 1–2, 168–179.
[3] I. Pawlikowska, *A method of solving functional equations on convex subsets of linear spaces*, Aequationes Math. **75** (2008), no. 1–2, 1–28.

László Székelyhidi: *On a generalization of the Heisenberg group*

In this talk, we present a generalization of the Heisenberg group and study its finite dimensional representations.

Tomasz Szostok: *The interplay between linear functional equations and inequalities*

We study connections between the equation

$$\sum_{i=1}^{n} a_i f(\alpha_i x + (1-\alpha_i)y) = 0$$

and the corresponding inequality

$$\sum_{i=1}^{n} a_i f(\alpha_i x + (1-\alpha_i)y) \geq 0.$$

Norbert Tóth: *An optimality condition for linear programs* (Joint work with Mihály Bessenyei)

Motivated by the well-known graphical method, we give a geometric characterization of those linear programs that have an optimal feasible solution. Our condition and approach rely on the notions and applications of the recession and normal cones. Among the applications, we present the strong duality theorem and revisit Farkas' lemma, as well.

References

[1] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, **54**, American Mathematical Society, Providence, RI, 2002.
[2] J.M. Borwein and A.S. Lewis, *Convex Analysis and Nonlinear Optimization. Theory and Examples*, 2nd ed., CMS Books in Mathematics, **3**, Springer, New York, 2006.
[3] J. Farkas, *Theorie der einfachen Ungleichungen*, J. Reine Angew. Math. **124** (1902), 1–27.
[4] R.T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series, **28**, Princeton University Press, Princeton, N.J., 1970.
[5] R.J. Vanderbei, *Linear Programming. Foundations and Extensions*, 4th ed., International Series in Operations Research & Management Science, **196**, Springer, New York, 2014.

Péter Tóth: *Strong geometric derivatives* (Joint work with Zoltán Boros)

Let I denote a nonempty open interval of the real line, and let us consider a function $f\colon I \to \mathbb{R}$. For $x \in I$ and $h \in \mathbb{R}$, we define the lower and upper strong geometric derivatives of f at the point x in the direction h by

$$\underline{D}^{\diamond}_{h} f(x) = \liminf_{\substack{y \to x \\ n \to \infty}} 2^n \left(f\left(y + \frac{h}{2^n}\right) - f(y) \right)$$

and

$$\overline{D}^{\diamond}_{h} f(x) = \limsup_{\substack{y \to x \\ n \to \infty}} 2^n \left(f\left(y + \frac{h}{2^n}\right) - f(y) \right),$$

respectively. We call f strongly geometrically differentiable if

$$\underline{D}^{\diamond}_{h} f(x) = \overline{D}^{\diamond}_{h} f(x) \in \mathbb{R}$$

holds for every $x \in I$ and $h \in \mathbb{R}$. We say that f has increasing strong geometric derivatives if

$$-\infty < \overline{D}^{\diamond}_{h} f(x) \le \underline{D}^{\diamond}_{h} f(y) < +\infty$$

holds for every $h > 0$ and $x, y \in I$ such that $x < y$. We give a characterization of these properties by the following decomposition theorems.

Theorem 1. *The function f is strongly geometrically differentiable if, and only if, there exist a continuously differentiable function $g\colon I \to \mathbb{R}$ and an additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + A(x)$ for every $x \in I$.*

Theorem 2. *The function f has increasing strong geometric derivatives if, and only if, there exist a convex function $g\colon I \to \mathbb{R}$ and an additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + A(x)$ for every $x \in I$.*

These investigations are motivated by similar concepts and related decomposition theorems, analogous to Theorem 1 ([1, 2]).

References

[1] Z. Boros, *Strongly $\mathbb{Q}$-differentiable functions*, Real Anal. Exchange **27** (2001/02), no. 1, 17–25.
[2] D. Broszka and Z. Grande, *On $\mathcal{I}$-differentiation*, Tatra Mt. Math. Publ. **35** (2007), 25–40.

Sebastian Wójcik: *Zero utility principle under uncertainty*

A process of insurance contract pricing consists on assigning to any risk, represented by a non-negative essentially bounded random variable on a given probability space, a non-negative real number, being a premium for the risk. In a literature, one can find various methods of insurance contracts pricing. In this talk, we consider the method, known as the zero utility principle, introduced by H. Bühlmann [1]. It presents the problem from the point of view of an insurance company, assuming that the premium for a given risk is determined in such a way that the company is indifferent between entering into contract and rejecting it.

We study the zero utility principle in the cumulative prospect theory (cf. [2]) under uncertainty. In this setting, the risks are represented by measurable function defined on a given measurable space $(S, \mathcal{F})$. The principle for a risk X is defined as a real number $H_{(u,\mu,\nu)}(X)$ satisfying the equation

$$E_{\mu\nu}[u(H_{(u,\mu,\nu)}(X) - X)] = 0,$$

where $E_{\mu\nu}$ is the Choquet integral with respect to the capacities $\mu, \nu \colon \mathcal{F} \to [0, 1]$.

We establish a necessary and sufficient condition for the existence of the principle. Furthermore, we characterize its important properties: the comparability, equality, positive homogeneity and comonotonic additivity.

References

[1] H. Bühlmann, *Mathematical Models in Risk Theory*, Springer-Verlag, Berlin, 1970.
[2] A. Tversky and D. Kahneman, *Advances in prospect theory: Cumulative representation of uncertainty*, J. Risk Uncertain. **5** (1992), no. 4, 297–323.

Thomas Zürcher: *Some information about my favourite Banach function space*

If you think about what properties make the Lebesgue spaces great and put them into a definition, then you get a pretty good idea what Banach function spaces are. One might think that Lebesgue spaces "are the best among them", but I do not share this opinion: there is a space that is even better than them. It is a part of a whole family of spaces, the so called *Lorentz spaces*. To introduce them, we need some definitions first. Assume that (X, μ) is a measure space and that $f \colon X \to [0, \infty]$ is measurable. Then we set

$$\omega_f(\alpha) := \mu(\{x \in X : f(x) > \alpha\}),$$

$$f^*(t) := \inf\{\alpha \geq 0 : \ \omega_f(\alpha) \leq t\}.$$

Let $p,q \in (0,\infty]$. The Lorentz space $L^{p,q}$ is the space of all measurable non-negative functions f such that

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty \left\{t^{1/p} f^*(t)\right\}^q \frac{dt}{t}\right)^{1/q}, & \text{if } 0<q<\infty \\ \sup_{0<t<\infty} \left\{t^{1/p} f^*(t)\right\}, & \text{if } q=\infty \end{cases}$$

is finite.

In the talk, we are interested in the case where $q = 1$.

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, **129**, Academic Press, Inc., Boston, MA, 1988.
[2] D.E. Edmunds and W.D. Evans, *Hardy Operators, Function Spaces and Embeddings*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.

2. Problems and Remarks

Problem. The Hermite–Hadamard inequality states that, for a convex function f, we have

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2}.$$

It means that, in some cases, the expression

$$\frac{1}{y-x}\int_x^y f(t)dt$$

may be considered as a mean of values $a = f\left(\frac{x+y}{2}\right)$ and $b = \frac{f(x)+f(y)}{2}$. This observation was an inspiration for the paper [1], where it was proved that the equation

$$\frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt = p\left(\frac{1}{y-x}\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right)\right)$$

may hold only for polynomials of degree not greater than 3. Thus our mean may be a weighted arithmetic mean only for such functions. In equations of this type, the integral may be replaced by the expression $F(y) - F(x)$, containing a new unknown function F. The equation obtained in this way may be considered without any regularity assumptions on the functions involved. It should be added here that a more general result may be found in [2] but the result from [1] is valid on for functions acting on rings.

Here, we pose the following problem. Find the solutions of a more general equation

$$\frac{1}{y-x}\int_x^y f(t)dt = \varphi^{-1}\left(\alpha\varphi\left(f\left(\frac{x+y}{2}\right)\right) + (1-\alpha)\varphi\left(\frac{f(x)+f(y)}{2}\right)\right).$$

Thus, we ask for which functions the integral $\frac{1}{y-x}\int_x^y f(t)dt$ is a weighted quasi arithmetic mean of $\frac{f(x)+f(y)}{2}$ and $f\left(\frac{x+y}{2}\right)$.

References

[1] B. Koclęga-Kulpa and T. Szostok, *On some functional equations connected to Hadamard inequalities*, Aequationes Math. **75** (2008), 119–129.

[2] P.K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore–New Jersey–London–Hong Kong, 1998.

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