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ON THE ZEROS OF POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \geq 0$. Then according to Eneström-Kakeya theorem all the zeros of $P(z)$ lie in $|z| \leq 1$. This result has been generalized in various ways (see [1, 3, 4, 6, 7]). In this paper we shall prove some generalizations of the results due to Aziz and Zargar [1, 2] and Nwaeze [7].

1. Introduction

In 1829, Cauchy [5] proved that if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n then all the zeros of $P(z)$ lie in

$$(1) \qquad |z| < 1 + M, \quad \text{where } M = \max\left\{\frac{|a_j|}{|a_n|} : j = 0, 1, 2, \ldots, n-1\right\}.$$

The following result due to Eneström and Kakeya [5] is well known in the theory of distribution of zeros of polynomials:

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$(2) \qquad a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \geq 0,$$

then $P(z)$ has all its zeros in $|z| \leq 1$.

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Aziz and Zargar [1] relaxed the hypothesis of inequality (2) in several ways and improved some of the bounds and among other things they proved the following result:

THEOREM A. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that either*

$$a_n \geq a_{n-2} \geq \ldots \geq a_3 \geq a_1 > 0$$

and

$$a_{n-1} \geq a_{n-3} \geq \ldots \geq a_2 \geq a_0 > 0, \quad \text{if } n \text{ is odd}$$

or

$$a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 > 0$$

and

$$a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1 > 0, \quad \text{if } n \text{ is even},$$

then all the zeros of $P(z)$ *lie in the circle*

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1}}{a_n}.$$

Aziz and Zargar [2] further relaxed the hypothesis and among other things proved the following result:

THEOREM B. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real coefficients such that for some positive numbers* k *and* η *with* $k \geq 1$ *and* $0 < \eta \leq 1$, $ka_n \geq a_{n-1} \geq \ldots \geq a_1 \geq \eta a_0 \geq 0$, *then all the zeros of* $P(z)$ *lie in the closed disk*

$$|z + k - 1| \leq \frac{ka_n + 2a_0(1-\eta)}{a_n}.$$

Nwaeze [7] proved the following result:

THEOREM C. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that for some real numbers* λ *and* ρ, $\lambda + a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 - \rho$, *then all the zeros of polynomial* $P(z)$ *lie in*

$$\left| z + \frac{\lambda}{a_n} \right| \leq \frac{1}{|a_n|} \Big\{ a_n + \lambda - a_0 + \rho + |\rho| + |a_0| \Big\}.$$

In this paper we shall present some extensions of the above results.

2. Main Results

Theorem 1. *If* $P(z)=\sum_{j=0}^{n}a_jz^j$, *where* $a_j=\alpha_j+i\beta_j, \alpha_j, \beta_j\in\mathbb{R}$, *is a polynomial of degree* n *such that for some real numbers* κ, λ, τ *and* ρ,

$$\lambda+\alpha_n\geq\alpha_{n-1}\geq\ldots\geq\alpha_1\geq\alpha_0-\rho$$

and

$$\kappa+\beta_n\geq\beta_{n-1}\geq\ldots\geq\beta_1\geq\beta_0-\tau,$$

then all the zeros of polynomial $P(z)$ *lie in*

$$\Big|z+\frac{\lambda+i\kappa}{a_n}\Big|\leq\frac{1}{|a_n|}\Big\{\alpha_n+\beta_n+\lambda+\kappa-(\alpha_0+\beta_0)+\tau+\rho+|\tau|+|\rho|+|\alpha_0|+|\beta_0|\Big\}.$$

If we take $\kappa=\tau=0$ in Theorem 1, we get the following result:

Corollary 1. *If* $P(z)=\sum_{j=0}^{n}a_jz^j$, *where* $a_j=\alpha_j+i\beta_j$, $\alpha_j, \beta_j\in\mathbb{R}$, *is a polynomial of degree* n, *such that for some real numbers* λ *and* ρ,

$$\lambda+\alpha_n\geq\alpha_{n-1}\geq\ldots\geq\alpha_1\geq\alpha_0-\rho, \quad \textit{and} \quad \beta_n\geq\beta_{n-1}\geq\ldots\geq\beta_1\geq\beta_0,$$

then all the zeros of polynomial $P(z)$ *lie in*

$$\Big|z+\frac{\lambda}{a_n}\Big|\leq\frac{1}{|a_n|}\Big\{\alpha_n+\beta_n+\lambda-(\alpha_0+\beta_0)+\rho+|\rho|+|\alpha_0|+|\beta_0|\Big\}.$$

Remark. If we take $\beta_j=0$, $j=0,1,\ldots,n$ in Corollary 1, we get Theorem C.

If we take $\lambda=(k-1)\alpha_n$ and $\rho=(1-\eta)\alpha_0$ in Corollary 1, we get the following result:

Corollary 2. *If* $P(z)=\sum_{j=0}^{n}a_jz^j$, *where* $a_j=\alpha_j+i\beta_j, \alpha_j, \beta_j\in\mathbb{R}$, *is a polynomial of degree* n *such that for some positive numbers* $k\geq 1$ *and* η *with* $0<\eta\leq 1$,

$$k\alpha_n\geq\alpha_{n-1}\geq\ldots\geq\alpha_1\geq\eta\alpha_0, \quad \textit{and} \quad \beta_n\geq\beta_{n-1}\geq\ldots\geq\beta_1\geq\beta_0,$$

then all the zeros of polynomial $P(z)$ *lie in*

$$|z+k-1|\leq\frac{1}{|a_n|}\Big\{k\alpha_n-\eta\alpha_0+\beta_n-\beta_0+(2-\eta)|\alpha_0|+|\beta_0|\Big\}.$$

REMARK. If we take $\beta_j = 0, j = 0, 1 \ldots, n$ and $\alpha_0 > 0$ in Corollary 2, we get Theorem B.

THEOREM 2. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients, such that for some positive numbers λ, κ, ρ and τ*

$$\lambda + a_n \geq a_{n-2} \geq \ldots \geq a_3 \geq a_1 - \tau$$

and

$$\kappa + a_{n-1} \geq a_{n-3} \geq \ldots \geq a_2 \geq a_0 - \rho, \quad \text{if } n \text{ is odd}$$

or

$$\lambda + a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 - \tau$$

and

$$\kappa + a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1 - \rho, \quad \text{if } n \text{ is even,}$$

then all the zeros of polynomial $P(z)$ lie in

$$|z + \frac{a_{n-1}}{a_n}| \leq \frac{1}{|a_n|}\Big(a_n + a_{n-1} - a_1 - a_0 + 2(\tau + \rho + \lambda + \kappa) + |a_1| + |a_0|\Big).$$

If we assume $a_0, a_1 > 0$, we get the following corollary:

COROLLARY 3. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients and $a_0, a_1 > 0$, such that for some positive numbers λ, κ, ρ and τ*

$$\lambda + a_n \geq a_{n-2} \geq \ldots \geq a_3 \geq a_1 - \tau$$

and

$$\kappa + a_{n-1} \geq a_{n-3} \geq \ldots \geq a_2 \geq a_0 - \rho, \quad \text{if } n \text{ is odd}$$

or

$$\lambda + a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 - \tau$$

and

$$\kappa + a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1 - \rho, \quad \text{if } n \text{ is even,}$$

then all the zeros of polynomial $P(z)$ lie in

$$|z + \frac{a_{n-1}}{a_n}| \leq \frac{1}{|a_n|}\Big(a_n + a_{n-1} + 2(\tau + \rho + \lambda + \kappa)\Big).$$

REMARK. If we take $\lambda = \kappa = \tau = \rho = 0$ in Corollary 3, we get Theorem A.

Examples

EXAMPLE 1. Let

$$P(z) = (8+7i)z^5 + (9+8i)z^4 + (4+7i)z^3 + (3+5i)z^2 + (2+3i)z + 1 + 2i.$$

Here the coefficients are $\alpha_5 = 8$, $\alpha_4 = 9$, $\alpha_3 = 4$, $\alpha_2 = 3$, $\alpha_1 = 2$, $\alpha_0 = 1$, $\beta_5 = 7$, $\beta_4 = 8$, $\beta_3 = 7$, $\beta_2 = 5$, $\beta_1 = 3$ and $\beta_0 = 2$.

Theorems A, B, and C are not applicable to this example, but we can apply Theorem 1. Taking $\lambda = 1$, $\kappa = 1$, $\rho = 0$ and $\tau = 0$, Theorem 1 locates the zeros of $P(z)$ in the region $|z + \frac{15+i}{113}| < 1.6$, which is better than the bound given by (1), i.e, $|z| < 2.13$. In fact the region $|z + \frac{15+i}{113}| < 1.6$ is contained in the region $|z| < 2.13$.

EXAMPLE 2. Let

$$P(z) = 40z^5 + 5z^4 + 41z^3 + 6z^2 + 30z - 1.$$

Here the coefficients are $a_5 = 40$, $a_4 = 5$, $a_3 = 41$, $a_2 = 6$, $a_1 = 30$, $a_0 = -1$.

Theorems A, B, C and 1 are not applicable to this example, but we can apply Theorem 2. Taking $\lambda = 1$, $\kappa = 1$, $\rho = 0$ and $\tau = 0$, Theorem 2 gives the region containing the zeros as $|z + \frac{5}{40}| \leq 1.275$, whereas Cauchy's bound (given by (1)) is $|z| < 2.025$. Thus the bound given by Theorem 2 is better than the bound given by (1). In fact $\{z : |z + \frac{5}{40}| \leq 1.275\} \subset \{z : |z| < 2.025\}$.

Proofs of Theorems

PROOF OF THEOREM 1. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0 \\
&\quad + i[(\beta_n - \beta_{n-1})z^n + \ldots + (\beta_1 - \beta_0)z + \beta_0] \\
&= -(a_n z + \lambda + i\kappa)z^n + (\alpha_n + \lambda - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\
&\quad + \ldots + (\alpha_1 - (\alpha_0 - \rho))z - \rho z + \alpha_0 + i[(\beta_n + \kappa - \beta_{n-1})z^n \\
&\quad + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots + (\beta_1 - (\beta_0 - \tau))z - \tau z + \beta_0] \\
&= -z^n(a_n z + \lambda + i\kappa) + q(z)
\end{aligned}$$

where

$$q(z) = (\alpha_n + \lambda - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots + (\alpha_1 - (\alpha_0 - \rho))z - \rho z + \alpha_0 + i[(\beta_n + \kappa - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots + (\beta_1 - (\beta_0 - \tau))z - \tau z + \beta_0].$$

Now, for $|z| = 1$, we have

$$\begin{aligned} |q(z)| &\leq |\alpha_n + \lambda - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| \\ &\quad + \ldots + |\alpha_1 - \alpha_0 + \rho| + |\rho| + |\alpha_0| + |\beta_n + \kappa - \beta_{n-1}| \\ &\quad + |\beta_{n-1} - \beta_{n-2}| + \ldots + |\beta_1 - \beta_0 + \tau| + |\tau| + |\beta_0| \\ &= \alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|. \end{aligned}$$

Since this is true for all complex numbers with unit modulus, then for $|z| = 1$,

$$|z^n q(1/z)| \leq \alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|.$$

Also the function $G(z) = z^n q(1/z)$ is analytic in $|z| \leq 1$. Hence, by maximum modulus theorem, for $|z| \leq 1$, we have

$$|q(1/z)| \leq \frac{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|}{|z|^n}.$$

Replacing z by $1/z$, we get for $|z| \geq 1$

$$|q(z)| \leq \Big\{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\Big\}|z|^n.$$

Now, for $|z| \geq 1$, we get

$$\begin{aligned} |F(z)| &= |-z^n(a_n z + \lambda + i\kappa) + q(z)| \\ &\geq |z^n(a_n z + \lambda + i\kappa| - |q(z)| \\ &\geq |z^n||a_n z + \lambda + i\kappa| - \Big\{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) \\ &\quad + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\Big\}|z|^n \end{aligned}$$

$$\implies |F(z)| \geq |z^n|\Big[|a_n z + \lambda + i\kappa| - \Big\{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\Big\}\Big] > 0,$$

if and only if

$$|a_n z + \lambda + i\kappa| > \{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\}$$

or

$$\Big|z + \frac{\lambda + i\kappa}{a_n}\Big| > \frac{1}{|a_n|}\Big\{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\Big\}.$$

Therefore all the zeros of $F(z)$, and hence of $P(z)$, whose modulus is greater or equal to 1 lie in

$$\Big|z + \frac{\lambda + i\kappa}{a_n}\Big| \leq \frac{1}{|a_n|}\Big\{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\Big\}.$$

Since any polynomial is an analytic function in $|z| \leq 1$ and by maximum modulus theorem it attains its maximum on the boundary $|z| = 1$ (in our case the polynomial may be taken as $z + \frac{\lambda + i\kappa}{a_n}$). It follows that all the zeros whose modulus is less than 1 lie in

$$\Big|z + \frac{\lambda + i\kappa}{a_n}\Big| \leq \frac{1}{|a_n|}\Big\{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\Big\}.$$

Therefore all the zeros of $P(z)$ lie in

$$\Big|z + \frac{\lambda + i\kappa}{a_n}\Big| \leq \frac{1}{|a_n|}\Big\{\alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0|\Big\}. \quad \square$$

Proof of Theorem 2. Let n be odd. Consider the polynomial

$$F(z) = (1 - z^2)P(z).$$

Then

$$\begin{aligned}
|F(z)| &= \Big| - a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} \\
&\qquad + \ldots + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0\Big| \\
&= \Big| - (a_n z + a_{n-1})z^{n+1} + (a_n + \lambda - a_{n-2})z^n - \lambda z^n \\
&\qquad + (a_{n-1} + \kappa - a_{n-3})z^{n-1} - \kappa z^{n-1} + \ldots + (a_3 - a_1 + \tau)z^3 - \tau z^3 \\
&\qquad + (a_2 - a_0 + \rho)z^2 - \rho z^2 + a_1 z + a_0\Big| \\
&\geq |z|^n \Big\{ |a_n z + a_{n-1}||z| - \Big(|a_n + \lambda - a_{n-2}| + |\lambda| \\
&\qquad + \frac{|a_{n-1} + \kappa - a_{n-3}|}{|z|} + \frac{|\kappa|}{z} + \ldots + \frac{|a_3 - a_1 + \tau|}{z^{n-3}} + \frac{|\tau|}{z^{n-3}} \\
&\qquad + \frac{|a_2 - a_0 + \rho|}{|z|^{n-2}} + \frac{|\rho|}{|z|^{n-2}} + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big)\Big\}.
\end{aligned}$$

Now, for $|z| \geq 1$, by using hypothesis we get

$$\begin{aligned}
|F(z)| &\geq |z|^n \Big\{ |a_n z + a_{n-1}||z| - \Big[a_n + \lambda - a_{n-2} + \lambda + a_{n-1} + \kappa - a_{n-3} \\
&\qquad + \kappa + \ldots + a_3 - a_1 + \tau + \tau + a_2 - a_0 + \rho + \rho + |a_1| + |a_0|\Big]\Big\} \\
&\geq |z|^n \Big\{ |a_n z + a_{n-1}| - \Big[a_n - a_{n-2} + a_{n-1} - a_{n-3} + a_{n-2} - a_{n-4} \\
&\qquad + \ldots + a_3 - a_1 + 2(\tau + \rho + \lambda + \kappa) + a_2 - a_0 + |a_1| + |a_0|\Big]\Big\} \\
&> 0,
\end{aligned}$$

if and only if

$$|a_n z + a_{n-1}| > \Big[a_n + a_{n-1} - a_1 - a_0 + 2(\tau + \rho + \lambda + \kappa) + |a_1| + |a_0|\Big].$$

Thus all the zeros of $F(z)$ whose modulus is greater than or equal to 1 lie in

$$|z + \frac{a_{n-1}}{a_n}| \leq \frac{1}{|a_n|}\Big(a_n + a_{n-1} - a_1 - a_0 + 2(\tau + \rho + \lambda + \kappa) + |a_1| + |a_0|\Big).$$

Since any polynomial is an analytic function in $|z| \leq 1$ and by maximum modulus theorem it attains its maximum on the boundary $|z| = 1$ (in our case the polynomial may be taken as $z + \frac{\lambda + i\kappa}{a_n}$). It follows that all the zeros whose modulus is less than 1 lie in

$$|z + \frac{a_{n-1}}{a_n}| \leq \frac{1}{|a_n|}\Big(a_n + a_{n-1} - a_1 - a_0 + 2(\tau + \rho + \lambda + \kappa) + |a_1| + |a_0|\Big).$$

Therefore all the zeros of $F(z)$ of odd degree lie in

$$|z + \frac{a_{n-1}}{a_n}| \leq \frac{1}{|a_n|}\Big(a_n + a_{n-1} - a_1 - a_0 + 2(\tau + \rho + \lambda + \kappa) + |a_1| + |a_0|\Big).$$

Since all the zeros of $P(z)$ are also zeros of $F(z)$, then all the zeros of $P(z)$ lie in the disk defined above. This completes the proof of the theorem for odd n. The proof for even n follows in the same way. □

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