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SUGENO INTEGRAL FOR HERMITE–HADAMARD INEQUALITY AND QUASI-ARITHMETIC MEANS

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Abstract. In this paper, we present the Sugeno integral of Hermite–Hadamard inequality for the case of quasi-arithmetically convex (q-ac) functions which acts as a generator for all quasi-arithmetic means in the frame work of Sugeno integral.

1. Introduction

The Hermite–Hadamard inequality is the first fundamental result for convex functions with a natural geometrical interpretation and many applications, has attracted and continues to attract much interest in elementary mathematics. Many mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions as seen in [2, 3, 4, 5, 6, 7, 8, 9, 11] etc.

The classical convexity is defined as follows:

Definition 1.1. A function $f\colon I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is a real interval, is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

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holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality is reversed f is said to be concave.

The classical Hermite–Hadamard inequality provides estimates of the mean value of a convex function $f\colon [a, b] \to \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

We also have the following Sugeno integral variant of the inequality (1.1).

Theorem 1.2 ([6]). *Let $f\colon [a, b] \subset (0, \infty) \to [0, +\infty)$ be a convex function with $f(a) \neq f(b)$. Then*

$$(s)\int_a^b f d\mu \le \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a, b] \cap \left\{x \ge \frac{\alpha(b-a)+af(a)-bf(b)}{f(b)-f(a)}\right\}\right)\right)$$

where $\Gamma = [\min(f(b), f(a)), \max(f(b), f(a))]$.

In the present work we formulate results connected to the Hermite–Hadamard inequality for quasi-arithmetic means and its Sugeno integral variant. This yields a generator for all the quasi-arithmetic means, for example see results [6].

Let us start with recalling the notion and properties of Sugeno integral.

Suppose (X, Σ, μ) is a fuzzy measure space and $\mathbf{F}$ is the class of all finite non-negative measurable functions defined on (X, Σ, μ). Then for any $f \in \mathbf{F}$, we write $f_\alpha = \{x : f(x) \ge \alpha\}$ for $\alpha \in [0, \infty]$.

Definition 1.3 (Generalized Sugeno integral). The *generalized Sugeno integral* of $f \in \mathbf{F}$ on $A \in \Sigma$ is defined as

$$\int_{\circ, A} f d\mu = \sup_{t\ge 0}(t \circ \mu(A \cap f_t)), \tag{1.2}$$

where μ is a monotone measure on Σ and $\circ$ is a non-decreasing binary map.

Commonly encountered examples of the generalized Sugeno integral include:

(1) the Sugeno integral

$$\int_A f d\mu = \sup_{t\ge 0}(t \wedge \mu(A \cap f_t)), \tag{1.3}$$

(2) the Shilkret integral

$$\int_A f d\mu = \sup_{t \geq 0}(t.\mu(A \cap f_t)),$$

(3) the q-integral and the semi-normed fuzzy integral.
Here and subsequently, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

Definition 1.4 (Sugeno integral [14]). Let μ be a fuzzy measure. If $f \in \mathbf{F}$ and $A \in \Sigma$, then the *Sugeno integral* of f on A with respect to the fuzzy measure μ is defined by

$$\int_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap f_\alpha)], \tag{1.4}$$

where $\bigvee$ denotes the operation sup, or upper bound. If $A = X$ then

$$\int_X f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(f_\alpha)].$$

Example 1.5. Let $X = [0, 1]$, $\mu = m^2$, where m is the Lebesgue measure, $f(x) = x/2$. Then

$$f_\alpha = \{x : f(x) \geq \alpha\} = [2\alpha, 1].$$

We only need to consider $\alpha \in [0, \frac{1}{2})$. So we have

$$\int_A f d\mu = \sup_{\alpha \in [0, \frac{1}{2})} [\alpha \wedge (1 - 2\alpha)^2].$$

Note that $\alpha \mapsto (1 - 2\alpha)^2$ is a decreasing continuous function whenever $\alpha \in [0, \frac{1}{2})$. Hence, the supremum will be attained at the point which is the solution of $\alpha = (1 - 2\alpha)^2$, that is, at $\alpha = \frac{1}{4}$.

Consequently, we have

$$\int_A f d\mu = \frac{1}{4}.$$

Let us enumerate some properties of the Sugeno integral which will be useful in the sequel.

PROPOSITION 1.6 ([14]). *If μ is a fuzzy measure on X and $f, g \in \boldsymbol{F}, \alpha \in [0, \infty]$ then:*

1. $\int_A f d\mu \leq \mu(A) \iff \mu(A \cap f_\alpha) \leq \mu(A), \alpha \geq 0.$
2. *If $\mathbb{1}_A$ is the characteristic function of A then $\int_X f \mathbb{1}_A d\mu = \int_A f d\mu$.*
3. $\int_A f d\mu > \alpha \iff \exists\, \beta > \alpha$ *such that* $\mu(A \cap f_\beta) > \alpha$.
4. *If $\mu(A) < \infty$, then $\int_A f d\mu \geq \alpha \iff \mu(A \cap f_\alpha) \geq \alpha$.*
5. *If $\mu(A) < \infty$, then $\int_A f d\mu \leq \alpha \iff \mu(A \cap f_\alpha) \leq \alpha$.*
6. *If $f_1 \leq f_2$ then $\int_A f_1 d\mu \leq \int_A f_2 d\mu$.*
7. $\int_A a d\mu = a \wedge \mu(A)$ *for any constant $a \in [0, \infty]$.*

REMARK 1.7. Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap f_\alpha)$. Then, due to 4. and 5. of Proposition 1.6, we have that $F(\alpha) = \alpha \Rightarrow \int_A f d\mu = \alpha$. Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

Now let us recall some classical results on quasi-arithmetic means and Hermite–Hadamard inequalities for them.

DEFINITION 1.8. Let $I \subset \mathbb{R}$ be an interval, and let $\phi\colon I \to \mathbb{R}$ and $\psi\colon J \to \mathbb{R}$ be strictly monotone and continuous functions. We say that $f\colon I \to \mathbb{R}$ is (ϕ, ψ)-*quasi-arithmetically convex* ((ϕ, ψ)-q-ac) if $f(I) \subset J$ and the following inequality

$$f(M_\phi(x, y, t)) \leq M_\psi(f(x), f(y), t) \tag{1.5}$$

holds for all $x, y \in I, t \in [0, 1]$. Here $M_\phi : I \times I \times [0, 1] \to I$ is defined by

$$M_\phi(x, y, t) := \phi^{-1}\left((1 - t)\phi(x) + t\phi(y)\right) \tag{1.6}$$

and $M_\psi : J \times J \times [0, 1] \to J$ is defined analogously.

REMARK 1.9. Let us note that we may assume without loss of generality that the generator ϕ is strictly increasing.

Functions $M_\phi\colon I \times I \times [0, 1] \to I$ and $M_\psi\colon J \times J \times [0, 1] \to J$ are called quasi-arithmetic means. Let us note that in particular we have $M_\phi\left(x, y, \frac{1}{2}\right) := \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)$. In particular, taking as $\phi\colon I \to \mathbb{R}$ the identity id_I and as ψ the id_J, we obtain

$$M_\phi(x, y, t) = (1 - t)x + ty \quad \text{and} \quad M_\psi(a, b, t) = (1 - t)a + tb,$$

in other words, both are linear means, and (1.5) becomes the classical convexity.

Using the hyperbolic function $\phi : (0,\infty) \to \mathbb{R}$ given by $\phi(x) = \frac{1}{x}$ in (1.6), we get the generalized harmonic mean

$$H(x,y,t) := \left((1-t)x^{-1} + ty^{-1}\right)^{-1} = \frac{xy}{tx+(1-t)y}.$$

And using the logarithmic function $\phi : (0,\infty) \to \mathbb{R}$ given by $\phi(x) = \ln x$ in (1.6), we obtain the generalized geometric mean

$$G(x,y,t) = x^{1-t}y^{t}.$$

Let us start with the following.

PROPOSITION 1.10 ([1]). *Let $I, J \subset \mathbb{R}$ be some intervals, and let $\phi\colon I \to \mathbb{R}$, $\psi\colon J \to \mathbb{R}$, be continuous and strictly monotone functions. Then $f\colon I \to \mathbb{R}$ is (ϕ,ψ)-qa-c if and only if $g = \psi\circ f\circ\phi^{-1}\colon \phi(I) \to \mathbb{R}$ is convex.*

We can express or compose the Borel probability measure in terms of a derivative and a measure. This gives us a special case of Theorem 1 in [11] as seen in the theorem below.

THEOREM 1.11 ([11]). *Let $I, J \subset \mathbb{R}$ be intervals, $\phi\colon I \to \mathbb{R}$ be a differentiable and strictly monotone function and let $\psi\colon J \to \mathbb{R}$ be strictly monotone and continuous. Further, let $f\colon I \to \mathbb{R}$ be a (ϕ,ψ)-qa-c function. Then the following inequalities hold*

$$\begin{aligned} f\left(M_\phi\left(x,y,\frac{1}{2}\right)\right) &\le \psi^{-1}\left(\frac{1}{\phi(y)-\phi(x)}\int_x^y \left(\psi\circ f\right)(u)\phi'(u)du\right) \\ &\le M_\psi\left(f(x),f(y),\frac{1}{2}\right), \end{aligned} \tag{1.7}$$

for all $x, y \in I, x \neq y$.

We have in particular the following

COROLLARY 1.12. *Let $I \subset \mathbb{R}$ be an interval, and let $\phi\colon I \to \mathbb{R}$ be a differentiable and strictly monotone function. Further, let $f\colon I \to \mathbb{R}$ be a (ϕ,id)-qa-c function. Then the following inequalities hold*

$$f\left(M_\phi\left(x,y,\frac{1}{2}\right)\right) \le \frac{1}{\phi(y)-\phi(x)}\int_x^y f(u)\phi'(u)du \le \frac{f(x)+f(y)}{2}, \tag{1.8}$$

for all $x, y \in I, x \neq y$.

2. Main results

In this section we extend the results in Theorem 1.11 by their Sugeno counterpart.

THEOREM 2.1. *Let M_ϕ, M_ψ be quasi-arithmetic means, generated by ϕ and ψ, respectively. Suppose that $f\colon \mathbb{R} \to [0,\infty)$ is (ϕ,ψ)-qa-c, i.e. for every $x, y \in I$, and $t \in [0,1]$ we have*

$$f\left(\phi^{-1}\left((1-t)\phi(x)+t\phi(y)\right)\right) \leq \psi^{-1}\left((1-t)\psi(f(x))+t\psi(f(y))\right).$$

If $f \in \boldsymbol{F}$ then

$$(2.1)\quad (s)\int_a^b f d\mu \leq \begin{cases} \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b]\cap\{x \geq Q(\alpha)\}\right)\right), & f(a) < f(b),\\ f(a)\wedge\mu([a,b]), & f(a) = f(b),\\ \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b]\cap\{x \leq Q(\alpha)\}\right)\right), & f(a) > f(b),\end{cases}$$

where $\Gamma = \operatorname{conv}\{f(a), f(b)\}$, and

$$Q(\alpha) := \phi^{-1}\left(\frac{\psi(\alpha)(\phi(b)-\phi(a))+\phi(a)\psi(f(b))-\phi(b)\psi(f(a))}{\psi(f(b))-\psi(f(a))}\right).$$

PROOF. Let $t_x = \frac{\phi(x)-\phi(a)}{\phi(b)-\phi(a)}$. Then

$$\phi(x) = (1-t_x)\phi(a) + t_x\phi(b).$$

Thus

$$\begin{aligned} f(x) &= f(\phi^{-1}(\phi(x)) = f(\phi^{-1}((1-t_x)\phi(a)+t_x\phi(b))) \\ &= f(M_\phi(a,b,t_x)) \leq M_\psi(f(a),f(b),t_x) \\ &= \psi^{-1}((1-t_x)\psi(f(a))+t_x\psi(f(b))) =: h(x). \end{aligned}$$

Hence, by 6. of Proposition 1.6 we get

$$(s)\int_a^b f d\mu \leq (s)\int_a^b \psi^{-1}((1-t_x)\psi(f(a))+t_x\psi(f(b))d\mu =: (s)\int_a^b h d\mu.$$

So we have

$$(s)\int_a^b f d\mu \le (s)\int_a^b h d\mu = \bigvee_{\alpha\ge 0}\left(\alpha\wedge\mu\left([a,b]\cap h_\alpha\right)\right), \tag{2.2}$$

where $h_\alpha = \{x \in [a,b] : h(x) \ge \alpha\}$. Consider the following three cases:

(i) $f(a) = f(b)$,
(ii) $f(a) < f(b)$,
(iii) $f(a) > f(b)$.

In case (i), we get $h = \text{const} = f(a)$. Then $(s)\int_a^b h d\mu = f(a) \wedge \mu([a,b])$.

Let us consider case (ii). First of all, we notice that $h([a,b]) = [f(a), f(b)]$. So, when we deal with $\alpha \in [0, f(a))$ we get

$$h_\alpha = \{x \in [a,b] : h(x) \ge \alpha\} = [a,b],$$

and consequently

$$\mu\left([a,b]\cap h_\alpha\right) = \mu([a,b]),$$

which implies

$$\bigvee_{0\le\alpha\le f(a)}\left(\alpha\wedge\mu\left([a,b]\cap h_\alpha\right)\right) = f(a)\wedge\mu([a,b]).$$

On the other hand, if $\alpha > f(b)$ then $H_\alpha = \emptyset$, and consequently $\mu([a,b]\cap H_\alpha) = 0$, which implies

$$\bigvee_{f(b)<\alpha}\left(\alpha\wedge\mu\left([a,b]\cap h_\alpha\right)\right) = 0.$$

We now obtain

$$\bigvee_{\alpha\ge 0}\left(\alpha\wedge\mu\left([a,b]\cap h_\alpha\right)\right) = \bigvee_{\alpha\in\Gamma}\left(\alpha\wedge\mu\left([a,b]\cap h_\alpha\right)\right),$$

which proves our assertion about Γ in case (ii), and we can replace (2.2) with

$$(s)\int_a^b f d\mu \le (s)\int_a^b h d\mu = \bigvee_{\alpha\in\Gamma}\left(\alpha\wedge\mu\left([a,b]\cap h_\alpha\right)\right). \tag{2.3}$$

Next is to find the formula (2.1). Assume that $f(a) < f(b)$ (the proof in the case $f(a) > f(b)$ is analogous):

$$\begin{aligned} h(x) \ge \alpha &\iff \psi^{-1}((1-t_x)\psi(f(a)) + t_x\psi(f(b)) \ge \alpha \\ &\iff x \ge \phi^{-1}\left(\frac{\psi(\alpha)(\phi(b)-\phi(a)) + \phi(a)\psi(f(b)) - \phi(b)\psi(f(a))}{\psi(f(b)) - \psi(f(a))}\right) \\ &\iff x \ge Q(\alpha). \end{aligned}$$

□

Corollary 2.2. *Let $f\colon \mathbb{R} \supset I \to [0,\infty)$ be a qa-c function satisfying*

$$f\left(\phi^{-1}\left((1-t)\phi(x) + t\phi(y)\right)\right) \le (1-t)f(x) + tf(y)$$

for every $x, y \in I$, and $t \in [0,1]$. If $f \in \boldsymbol{F}$ then

$$(2.4)\quad (s)\int_a^b f d\mu \le \begin{cases} \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b] \cap \{x \ge P(\alpha)\}\right)\right), & f(a) < f(b), \\ f(a) \wedge \mu([a,b]), & f(a) = f(b), \\ \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b] \cap \{x \le P(\alpha)\}\right)\right), & f(a) > f(b), \end{cases}$$

where $\Gamma = \operatorname{conv}\{f(a), f(b)\}$, *and*

$$P(\alpha) := \phi^{-1}\left(\frac{\alpha(\phi(b)-\phi(a)) + \phi(a)f(b) + \phi(b)f(a)}{f(b) - f(a)}\right).$$

Proof. Take $\psi = \mathrm{id}_{[0,\infty)}$ in the previous theorem. □

Example 2.3. From Theorem 2.1, we can obtain the Hermite–Hadamard inequality for Sugeno integral for different (ϕ,ψ)-qa-means. In particular, considering $\phi(x) = x = \psi(x)$ in (2.1), we obtain

$$Q(\alpha) = \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)},$$

which is the linear mean.

Similarly, for $\phi(x) = \frac{1}{x} = \psi(x)$, we have

$$Q(\alpha) = \frac{\alpha ab(f(a) - f(b))}{(a-b)f(b)f(a) + \alpha(bf(a) - af(b))},$$

and this gives us Hermite–Hadamard inequality for Sugeno integral based on harmonic mean.

Next, for $\phi(x) = \ln(x) = \psi(x)$,

$$Q(\alpha) = \exp(R(\alpha)),$$

where

$$R(\alpha) = \left(\frac{\ln \alpha(\ln b - \ln a) + \ln a \ln f(b) - \ln b \ln f(a)}{\ln f(b) - \ln f(a)} \right).$$

Thus we obtain Hermite–Hadamard inequality for Sugeno integral based on geometric mean.

We may also consider different means for ϕ and ψ, in particular $\phi(x) = \frac{1}{x}$ and $\psi(x) = \ln x$. We obtain then

$$Q(\alpha) = \frac{ab(\ln f(a) - \ln f(b))}{\ln \alpha(a-b) + b \ln f(b) - a \ln f(a)},$$

and this gives Hermite–Hadamard inequality for Sugeno integral based on geometric-harmonic means.

In the same way we can obtain other H-H inequalities for Sugeno integral from (2.1).

Example 2.4. From Corollary 2.2, we can generate the Hermite–Hadamard inequality for Sugeno integral for different qa-means. In particular, putting $\phi(x) = x$ in (2.4), we obtain

$$P(\alpha) = \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)},$$

which is the linear mean.

Similarly, for $\phi(x) = \frac{1}{x}$, we have

$$P(\alpha) = \frac{ab\,(f(b) - f(a))}{\alpha(a-b) + bf(b) - af(a)}$$

and this gives us harmonic mean.

And for $\phi(x) = \ln(x)$, we obtain geometric mean with

$$P(\alpha) = \exp(Q(\alpha)),$$

where

$$Q(\alpha) = \left(\frac{\alpha(\ln a - \ln b) + f(b) \ln a - f(a) \ln b}{f(b) - f(a)} \right).$$

In the same way we can obtain other H-H inequalities for Sugeno integral from (2.4).

COROLLARY 2.5. *Consider a measure space (X, Σ, μ) where μ is a Lebesgue measure on $X = \mathbb{R}$, then from inequality (2.1) we obtain*

$$(s)\int_a^b f d\mu \leq \begin{cases} \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \left(b - Q(\alpha)\right)\right), & f(a) < f(b), \\ f(a) \wedge b - a, & f(a) = f(b), \\ \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \left(Q(\alpha) - a\right)\right), & f(a) > f(b), \end{cases} \tag{2.5}$$

where

$$Q(\alpha) := \phi^{-1}\left(\frac{\psi(\alpha)(\phi(b)-\phi(a)) + \phi(a)\psi(f(b)) - \phi(b)\psi(f(a))}{\psi(f(b)) - \psi(f(a))}\right).$$

We now compute (2.5) for particular cases of qa-means. Starting with linear mean where

$$Q(\alpha) = \left(\frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}\right),$$

consider the following cases:

(i) $f(a) = f(b)$,
(ii) $f(a) < f(b)$,
(iii) $f(a) > f(b)$.

Case (i) does not change. And for case (ii) the minimum between α and $b - Q(\alpha)$ is attained at their point of intersection due to being strictly monotonic. So,

$$\alpha = b - Q(\alpha) \iff \alpha = b - \frac{\alpha(a-b) + af(b) - bf(a)}{f(b) - f(a)}$$
$$\iff \alpha(f(b) - f(a) + a - b) = b(f(b) - f(a)) - af(b) + bf(a).$$

So

$$\alpha = \frac{b(f(b) - f(a)) + bf(a) - af(b)}{f(b) + a - b - f(a)}.$$

Taking into account Remark 1.7 and 1. of Proposition 1.6 we get

$$(s)\int_a^b f d\mu \leq \frac{b(f(b) - f(a)) + bf(a) - af(b)}{f(b) + a - b - f(a)} \wedge (b - a).$$

The proof for case (iii) is analogous.

Next is the harmonic mean, similarly like in the case of linear mean. We see that

$$\alpha = b - \frac{\alpha ab(f(a) - f(b))}{(a-b)f(a)f(b) + \alpha(bf(a) - af(b))},$$

so,

$$\alpha = \frac{-A \pm \sqrt{B^2 - 4AC}}{2A}, \tag{2.6}$$

where $A = bf(a) - af(b)$, $B = (a-b)f(b)f(a) - b^2 f(a) + abf(b) + abf(a) - abf(b)$, and $C = -b(a-b)f(b)f(a)$. From Remark 1.7 and 1. of Proposition 1.6 we have

$$(s)\int_a^b f d\mu \leq \alpha \wedge (b-a)$$

with α defined in (2.6). The proof for case (iii) is analogous. And case (i) is constant. In the same way we are able to estimate integral in (2.5) for other particular cases of qa-means.

References

[1] J. Aczél, *The notion of mean values*, Norske Vid. Selsk. Forh., Trondhjem **19** (1947), no. 23, 83–86.

[2] J. Caballero and K. Sadarangani, *Hermite–Hadamard inequality for fuzzy integrals*, Appl. Math. Comput. **215** (2009), no. 6, 2134–2138.

[3] P. Czinder and Zs. Páles, *An extension of the Hermite–Hadamard inequality and an application for Gini and Stolarsky means*, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), no. 2, Art. 42, 8 pp.

[4] S.S. Dragomir, *Two refinements of Hadamard's inequalities*, Zb. Rad. (Kragujevac) No. **11** (1990), 23–26.

[5] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, Science Direct Working Paper No S1574-0358(04)70845-X, 2003.

[6] A. Ebadian and M. Oraki, *Hermite–Hadamard inequality for Sugeno integral based on harmonically convex functions*, J. Comput. Anal. Appl. **29** (2021), no. 3, 532–543.

[7] D.H. Hong, *Equivalent conditions of Hermite–Hadamard type inequality for Sugeno integrals*, Int. J. Math. Anal. (Ruse) **10** (2016), no. 27, 1323–1331.

[8] O. Hutnik, *On Hadamard type inequalities for generalized weighted quasi-arithmetic means*, JIPAM. J. Inequal. Pure Appl. Math. **7** (2006), no. 3, Art. 96, 10 pp.

[9] İ. İşcan, *Hermite–Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat. **43** (2014), no. 6, 935–942.

[10] D.-Q. Li, X.-Q. Song, and T. Yue, *Hermite–Hadamard type inequality for Sugeno integrals*, Appl. Math. Comput. **237** (2014), 632–638.

[11] F.C. Mitroi and C.I. Spiridon, *Hermite–Hadamard type inequalities of convex functions with respect to a pair of quasi-arithmetic means*, Math. Rep. (Bucur.) **14(64)** (2012), no. 3, 291–295.

[12] C.P. Niculescu, *The Hermite–Hadamard inequality for log-convex functions*, Nonlinear Anal. **75** (2012), no. 2, 662–669.

[13] C.P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics, **23**, Springer, New York, 2006.

[14] Z.Y. Wang and G.J. Klir, *Fuzzy Measure Theory*, Plenum Press, New York, 1992.

[15] X.-M. Zhang, Y.-M. Chu, and X.-H. Zhang, *The Hermite–Hadamard type inequality of GA-convex functions and its applications*, J. Inequal. Appl. **2010**, Art. ID 507560, 11 pp.

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