

GRADIENT INEQUALITIES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR 

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H .

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M . Then

$$0 \leq -m\mathcal{D}'(w, \mu)(\delta) \leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \leq -M\mathcal{D}'(w, \mu)(\alpha),$$

where $\mathcal{D}'(w, \mu)(t)$ is the derivative of $\mathcal{D}(w, \mu)(t)$ as a function of $t > 0$.

If $f: [0, \infty) \rightarrow \mathbb{R}$ is operator monotone on $[0, \infty)$ with $f(0) = 0$, then

$$\begin{aligned} 0 &\leq \frac{m}{\delta^2} [f(\delta) - f'(\delta)\delta] \leq f(A)A^{-1} - f(B)B^{-1} \\ &\leq \frac{M}{\alpha^2} [f(\alpha) - f'(\alpha)\alpha]. \end{aligned}$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{D}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

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1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be *positive* (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions ([7], [6]), see for instance [1, p. 144-145]:

THEOREM 1. *A function $f: [0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda),$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$(1.1) \quad \int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.$$

A real valued continuous function f on an interval I is said to be *operator convex* (operator concave) on I if

$$(OC) \quad f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B)$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I . Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions ([1, p. 147]):

THEOREM 2. *A function $f: [0, \infty) \rightarrow \mathbb{R}$ is operator convex in $[0, \infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0)t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda),$$

where $c \geq 0$ and a positive measure μ on $[0, \infty)$ such that (1.1) holds.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda+t} d\lambda.$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln \left(\frac{u+t}{u+1} \right)$$

for all $u > 0$. By taking the limit over $u \rightarrow \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$(1.2) \quad \mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \quad t > 0,$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.2) exists for all $t > 0$. For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \quad t > 0.$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$(1.3) \quad t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0.$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$(1.4) \quad \ln t = (t-1) \mathcal{D}(w_{\ln})(t), \quad t > 0.$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\mu(\lambda),$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\lambda,$$

for $T > 0$.

From (1.3) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(T)$$

where $T > 0$ and from (1.4)

$$(T - 1)^{-1} \ln T = \mathcal{D}(w_{\ln})(T)$$

provided $T > 0$ and $T - 1$ is invertible.

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α , δ , m , M . In this paper we show among others that

$$0 \leq -m\mathcal{D}'(w, \mu)(\delta) \leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \leq -M\mathcal{D}'(w, \mu)(\alpha),$$

where $\mathcal{D}'(w, \mu)(t)$ is the derivative of $\mathcal{D}(w, \mu)(t)$ as a function of $t > 0$. Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

Let f be an operator convex function on I . For $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I , we consider the auxiliary function $\varphi_{(A,B)}: [0, 1] \rightarrow \mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t) := f((1-t)A + tB).$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}: [0, 1] \rightarrow \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \langle \varphi_{(A,B)}(t)x, x \rangle = \langle f((1-t)A + tB)x, x \rangle.$$

We have the following basic fact ([2]):

LEMMA 1. *Let f be an operator convex function on I . For any $A, B \in \mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $A, B \in \mathcal{SA}_I(H)$ and $x \in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0, 1]$.*

A continuous function $g: \mathcal{SA}_I(H) \rightarrow \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$(2.1) \quad \nabla g_A(B) := \lim_{s \rightarrow 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H).$$

If the limit (2.1) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I , by utilizing the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A, B] := \{(1-t)A + tB \mid t \in [0, 1]\}.$$

We observe that $A, B \in [A, B]$ and $[A, B] \subset \mathcal{SA}_I(H)$.

We also have ([2]):

LEMMA 2. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A, B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0, 1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A).$$

In particular,

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A)$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A).$$

and, see [2],

LEMMA 3. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A, B])$, then for $0 < t_1 < t_2 < 1$*

$$\nabla f_{(1-t_1)A+t_1B}(B-A) \leq \nabla f_{(1-t_2)A+t_2B}(B-A)$$

in the operator order.

In particular,

$$\nabla f_A(B-A) \leq \nabla f_{(1-t_1)A+t_1B}(B-A)$$

and

$$\nabla f_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A).$$

Also, we have

$$(2.2) \quad \nabla f_A(B-A) \leq \nabla f_{(1-t)A+tB}(B-A) \leq \nabla f_B(B-A)$$

for all $t \in (0, 1)$.

We have the following gradient inequalities:

LEMMA 4. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A, B])$, then*

$$(2.3) \quad \nabla f_B(B-A) \geq f(B) - f(A) \geq \nabla f_A(B-A).$$

PROOF. By the properties of Bochner integral, we have

$$\begin{aligned} f(B) - f(A) &= \varphi_{(A,B)}(1) - \varphi_{(A,B)}(0) = \int_0^1 \varphi'_{(A,B)}(t) dt \\ &= \int_0^1 \nabla f_{(1-t)A+tB}(B-A) dt. \end{aligned}$$

From (2.2) we have, by integration, that

$$\nabla f_A(B-A) \leq \int_0^1 \nabla f_{(1-t)A+tB}(B-A) dt \leq \nabla f_B(B-A),$$

and the inequality (2.3) is proved. $\square$

Let $T, S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$(2.4) \quad \nabla f_T(S) := \lim_{t \rightarrow 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = -T^{-1}ST^{-1}$$

for $T, S > 0$.

Using (2.4) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$(2.5) \quad D^{-1}(D-C)D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D-C)C^{-1}$$

for all $C, D > 0$. If

$$m \leq D - C \leq M$$

for some constants m, M , then

$$mD^{-2} \leq D^{-1}(D-C)D^{-1}$$

and

$$C^{-1}(D-C)C^{-1} \leq MC^{-2}$$

and by (2.5) we derive

$$mD^{-2} \leq C^{-1} - D^{-1} \leq MC^{-2}.$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2}.$$

We have the following lower and upper bounds for $\mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B)$ which is a nonnegative operator in the general case when $B - A \geq 0$.

THEOREM 3. *Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M . Then*

$$(2.6) \quad 0 \leq -m\mathcal{D}'(w, \mu)(\delta) \leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \leq -M\mathcal{D}'(w, \mu)(\alpha),$$

where $\mathcal{D}'(w, \mu)(t)$ is the derivative of $\mathcal{D}(w, \mu)(t)$ as a function of $t > 0$.

PROOF. We have

$$\mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) = \int_0^\infty w(\lambda) \left[(\lambda + A)^{-1} - (\lambda + B)^{-1} \right] d\mu(\lambda).$$

From (2.5) we get for $C = \lambda + A$ and $D = \lambda + B$ that

$$(2.7) \quad (\lambda + B)^{-1} (B - A) (\lambda + B)^{-1} \leq (\lambda + A)^{-1} - (\lambda + B)^{-1} \\ \leq (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1}$$

for all $\lambda \geq 0$.

If we multiply (2.7) by $w(\lambda) \geq 0$ and integrate over $d\mu(\lambda)$ we get

$$(2.8) \quad \int_0^\infty w(\lambda) (\lambda + B)^{-1} (B - A) (\lambda + B)^{-1} d\mu(\lambda) \\ \leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \\ \leq \int_0^\infty w(\lambda) (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\mu(\lambda).$$

Since $m \leq B - A \leq M$ hence

$$m(\lambda + B)^{-2} \leq (\lambda + B)^{-1} (B - A) (\lambda + B)^{-1},$$

which implies, by integration, that

$$(2.9) \quad \begin{aligned} m \int_0^\infty w(\lambda) (\lambda + B)^{-2} d\mu(\lambda) \\ \leq \int_0^\infty w(\lambda) (\lambda + B)^{-1} (B - A) (\lambda + B)^{-1} d\mu(\lambda). \end{aligned}$$

Also

$$(\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} \leq M (\lambda + A)^{-2},$$

which implies, by integration, that

$$(2.10) \quad \begin{aligned} \int_0^\infty w(\lambda) (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\mu(\lambda) \\ \leq M \int_0^\infty w(\lambda) (\lambda + A)^{-2} d\mu(\lambda). \end{aligned}$$

Since $B \leq \delta$, then $\lambda + B \leq \lambda + \delta$ for all $\lambda \geq 0$ which implies that $(\lambda + B)^{-1} \geq (\lambda + \delta)^{-1}$ and therefore $(\lambda + B)^{-2} \geq (\lambda + \delta)^{-2}$. Consequently

$$(2.11) \quad m \int_0^\infty w(\lambda) (\lambda + B)^{-2} d\mu(\lambda) \geq m \int_0^\infty w(\lambda) (\lambda + \delta)^{-2} d\mu(\lambda).$$

Also, since $A \geq \alpha > 0$, then $\lambda + A \geq \lambda + \alpha > 0$, which implies that $(\lambda + A)^{-1} \leq (\lambda + \alpha)^{-1}$, therefore $(\lambda + A)^{-2} \leq (\lambda + \alpha)^{-2}$ and

$$(2.12) \quad M \int_0^\infty w(\lambda) (\lambda + A)^{-2} d\mu(\lambda) \leq M \int_0^\infty w(\lambda) (\lambda + \alpha)^{-2} d\mu(\lambda).$$

From (2.8)–(2.12) we get

$$(2.13) \quad \begin{aligned} m \int_0^\infty w(\lambda) (\lambda + \delta)^{-2} d\mu(\lambda) &\leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \\ &\leq M \int_0^\infty w(\lambda) (\lambda + \alpha)^{-2} d\mu(\lambda). \end{aligned}$$

For $h \neq 0$ small,

$$\begin{aligned} \frac{\mathcal{D}(w, \mu)(t + h) - \mathcal{D}(w, \mu)(t)}{h} &= \frac{1}{h} \int_0^\infty \left(\frac{w(\lambda)}{t + h + \lambda} - \frac{w(\lambda)}{t + \lambda} \right) d\mu(\lambda) \\ &= - \int_0^\infty \frac{w(\lambda)}{(t + h + \lambda)(t + \lambda)} d\mu(\lambda). \end{aligned}$$

By taking the limit over $h \rightarrow 0$ and using the properties of limits and integrals, we get the derivative of $\mathcal{D}(w, \mu)$ as

$$(2.14) \quad \mathcal{D}'(w, \mu)(t) = - \int_0^\infty \frac{w(\lambda)}{(t + \lambda)^2} d\mu(\lambda) \leq 0, \quad t > 0.$$

From (2.13) and (2.14) we derive (2.6). □

We know that for $T > 0$, we have the operator inequalities

$$(2.15) \quad 0 < \|T^{-1}\|^{-1} \leq T \leq \|T\|.$$

Indeed, it is well known that, if $P \geq 0$, then

$$|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle$$

for all $x, y \in H$. Therefore, if $T > 0$, then

$$\begin{aligned} 0 &\leq \langle x, x \rangle^2 = \langle T^{-1}Tx, x \rangle^2 = \langle Tx, T^{-1}x \rangle^2 \\ &\leq \langle Tx, x \rangle \langle TT^{-1}x, T^{-1}x \rangle = \langle Tx, x \rangle \langle x, T^{-1}x \rangle \end{aligned}$$

for all $x \in H$. If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x \rangle \langle x, T^{-1}x \rangle \leq \langle Tx, x \rangle \sup_{\|x\|=1} \langle x, T^{-1}x \rangle = \langle Tx, x \rangle \|T^{-1}\|,$$

which implies the following operator inequality

$$\|T^{-1}\|^{-1} 1_H \leq T.$$

The second inequality in (2.15) is obvious.

COROLLARY 1. *If $A, B > 0$ and $B - A > 0$, then*

$$\begin{aligned} (2.16) \quad 0 &\leq - \left\| (B - A)^{-1} \right\|^{-1} \mathcal{D}'(w, \mu)(\|B\|) \leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \\ &\leq - \|B - A\| \mathcal{D}'(w, \mu) \left(\|A^{-1}\|^{-1} \right). \end{aligned}$$

PROOF. Since $A \geq \|A^{-1}\|^{-1} = \alpha > 0$, $\delta = \|B\| \geq B > 0$ and

$$0 < m = \|(B - A)^{-1}\|^{-1} \leq B - A \leq \|B - A\| = M,$$

then by (2.6) we get (2.16). $\square$

The case of operator monotone functions is as follows:

COROLLARY 2. Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α , δ , m , M . If $f: [0, \infty) \rightarrow \mathbb{R}$ is operator monotone on $[0, \infty)$, then

$$\begin{aligned} (2.17) \quad 0 &\leq \frac{m}{\delta^2} [f(\delta) - f(0) - f'(\delta)\delta] \\ &\leq f(A)A^{-1} - f(B)B^{-1} - f(0)(A^{-1} - B^{-1}) \\ &\leq \frac{M}{\alpha^2} [f(\alpha) - f(0) - f'(\alpha)\alpha]. \end{aligned}$$

If $f(0) = 0$, then

$$\begin{aligned} (2.18) \quad 0 &\leq \frac{m}{\delta^2} [f(\delta) - f'(\delta)\delta] \leq f(A)A^{-1} - f(B)B^{-1} \\ &\leq \frac{M}{\alpha^2} [f(\alpha) - f'(\alpha)\alpha]. \end{aligned}$$

PROOF. We have that

$$\frac{f(t) - f(0)}{t} - b = \int_0^\infty \frac{\lambda}{\lambda + t} d\mu(\lambda) = \mathcal{D}(\ell, \mu)(t), \quad t > 0$$

with $\ell(\lambda) = \lambda$, for some positive measure $\mu(\lambda)$ and nonnegative b . From this,

$$\mathcal{D}'(\ell, \mu)(t) = \frac{f'(t)t - f(t) + f(0)}{t^2}, \quad t > 0.$$

Then by (2.6) we get

$$\begin{aligned} 0 &\leq \frac{m}{\delta^2} [f(\delta) - f(0) - f'(\delta)\delta] \\ &\leq [f(A) - f(0)]A^{-1} - [f(B) - f(0)]B^{-1} \leq \frac{M}{\alpha^2} [f(\alpha) - f(0) - f'(\alpha)\alpha], \end{aligned}$$

which is equivalent to (2.17). $\square$

REMARK 1. If we write the inequality (2.18) for the operator monotone function $f(t) = t^r$, $r \in (0, 1]$, then we get the power inequalities

$$0 < (1-r) \delta^{r-2} m \leq A^{r-1} - B^{r-1} \leq (1-r) \alpha^{r-2} M,$$

provided that A, B satisfy the assumptions in Corollary 2.

We also have the logarithmic inequalities

$$\begin{aligned} 0 &\leq \frac{m}{\delta^2} \left[\ln(\delta + 1) - (\delta + 1)^{-1} \delta \right] \leq A^{-1} \ln(A + 1) - B^{-1} \ln(B + 1) \\ &\leq \frac{M}{\alpha^2} \left[\ln(\alpha + 1) - (\alpha + 1)^{-1} \alpha \right]. \end{aligned}$$

We also have:

COROLLARY 3. Let $A, B > 0$ and $B - A > 0$. If $f: [0, \infty) \rightarrow \mathbb{R}$ is operator monotone on $[0, \infty)$, then

$$\begin{aligned} 0 &\leq \frac{1}{\|B\|^2 \|(B-A)^{-1}\|} [f(\|B\|) - f(0) - f'(\|B\|) \|B\|] \\ &\leq f(A) A^{-1} - f(B) B^{-1} - f(0) (A^{-1} - B^{-1}) \\ &\leq \|B - A\| \|A^{-1}\|^2 \left[f(\|A^{-1}\|^{-1}) - f(0) - \frac{f'(\|A^{-1}\|^{-1})}{\|A^{-1}\|} \right]. \end{aligned}$$

If $f(0) = 0$, then

$$\begin{aligned} (2.19) \quad 0 &\leq \frac{1}{\|B\|^2 \|(B-A)^{-1}\|} [f(\|B\|) - f'(\|B\|) \|B\|] \\ &\leq f(A) A^{-1} - f(B) B^{-1} \\ &\leq \|B - A\| \|A^{-1}\|^2 \left[f(\|A^{-1}\|^{-1}) - \frac{f'(\|A^{-1}\|^{-1})}{\|A^{-1}\|} \right]. \end{aligned}$$

If we take $f(t) = t^r$, $r \in (0, 1]$ in (2.19), then we get the power inequalities

$$0 < \frac{(1-r) \|B\|^{r-2}}{\|(B-A)^{-1}\|} \leq A^{r-1} - B^{r-1} \leq (1-r) \|B - A\| \|A^{-1}\|^{2-r},$$

for $A, B > 0$ and $B - A > 0$.

We also have the logarithmic inequalities

$$\begin{aligned}
 0 &\leq \frac{1}{\|B\|^2 \|(B-A)^{-1}\|} \left[\ln(\|B\| + 1) - (\|B\| + 1)^{-1} \|B\| \right] \\
 &\leq A^{-1} \ln(A + 1) - B^{-1} \ln(B + 1) \\
 &\leq \|B - A\| \|A^{-1}\|^2 \left[\ln\left(\|A^{-1}\|^{-1} + 1\right) - \left(\|A^{-1}\|^{-1} + 1\right)^{-1} \|A^{-1}\|^{-1} \right].
 \end{aligned}$$

The case of operator convex functions is as follows:

COROLLARY 4. *Assume that A, B are as in Corollary 2. If $f: [0, \infty) \rightarrow \mathbb{R}$ is operator convex on $[0, \infty)$, then*

$$\begin{aligned}
 (2.20) \quad 0 &\leq \frac{2m}{\delta^2} \left(\frac{f(\delta) - f(0)}{\delta} - \frac{f'(\delta) + f'_+(0)}{2} \right) \\
 &\leq f(A) A^{-2} - f(B) B^{-2} - f(0) (A^{-2} - B^{-2}) \\
 &\quad - f'_+(0) (A^{-1} - B^{-1}) \\
 &\leq \frac{2M}{\alpha^2} \left(\frac{f(\alpha) - f(0)}{\alpha} - \frac{f'(\alpha) + f'_+(0)}{2} \right).
 \end{aligned}$$

If $f(0) = 0$, then

$$\begin{aligned}
 (2.21) \quad 0 &\leq \frac{2m}{\delta^2} \left(\frac{f(\delta)}{\delta} - \frac{f'(\delta) + f'_+(0)}{2} \right) \\
 &\leq f(A) A^{-2} - f(B) B^{-2} - f'_+(0) (A^{-1} - B^{-1}) \\
 &\leq \frac{2M}{\alpha^2} \left(\frac{f(\alpha)}{\alpha} - \frac{f'(\alpha) + f'_+(0)}{2} \right).
 \end{aligned}$$

PROOF. We have that

$$\frac{f(t) - f(0) - f'_+(0)t}{t^2} - c = \int_0^\infty \frac{\lambda}{\lambda + t} d\mu(\lambda) = \mathcal{D}(\ell, \mu)(t), \quad t \geq 0$$

with $\ell(\lambda) = \lambda$ for some positive measure $\mu(\lambda)$ and nonnegative c .

We have that

$$\begin{aligned}\mathcal{D}'(\ell, \mu)(t) &= \frac{(f'(t) - f'_+(0))t^2 - 2t(f(t) - f(0) - f'_+(0)t)}{t^4} \\ &= \frac{2}{t^2} \left(\frac{f'(t) + f'_+(0)}{2} - \frac{f(t) - f(0)}{t} \right).\end{aligned}$$

Since

$$\begin{aligned}\mathcal{D}(\ell, \mu)(A) - \mathcal{D}(\ell, \mu)(B) &= [f(A) - f(0) - f'_+(0)A]A^{-2} - [f(B) - f(0) - f'_+(0)B]B^{-2} \\ &= f(A)A^{-2} - f(B)B^{-2} - f(0)(A^{-2} - B^{-2}) - f'_+(0)(A^{-1} - B^{-1}),\end{aligned}$$

$$-m\mathcal{D}'(\ell, \mu)(\delta) = \frac{2m}{\delta^2} \left(\frac{f(\delta) - f(0)}{\delta} - \frac{f'(\delta) + f'_+(0)}{2} \right)$$

and

$$-M\mathcal{D}'(\ell, \mu)(\alpha) = \frac{2M}{\alpha^2} \left(\frac{f(\alpha) - f(0)}{\alpha} - \frac{f'(\alpha) + f'_+(0)}{2} \right),$$

hence by (2.6) we derive (2.20). $\square$

COROLLARY 5. *Let $A, B > 0$ and $B - A > 0$. If $f: [0, \infty) \rightarrow \mathbb{R}$ is operator convex on $[0, \infty)$, then*

$$\begin{aligned}0 &\leq \frac{2}{\|B\|^2 \|(B - A)^{-1}\|} \left(\frac{f(\|B\|) - f(0)}{\|B\|} - \frac{f'(\|B\|) + f'_+(0)}{2} \right) \\ &\leq f(A)A^{-2} - f(B)B^{-2} - f(0)(A^{-2} - B^{-2}) - f'_+(0)(A^{-1} - B^{-1}) \\ &\leq 2\|B - A\| \|A^{-1}\|^2 \\ &\quad \times \left(\|A^{-1}\| \left[f(\|A^{-1}\|^{-1}) - f(0) \right] - \frac{f'(\|A^{-1}\|^{-1}) + f'_+(0)}{2} \right).\end{aligned}$$

If $f(0) = 0$, then

$$\begin{aligned}
 (2.22) \quad 0 &\leq \frac{2}{\|B\|^2 \|(B-A)^{-1}\|} \left(\frac{f(\|B\|)}{\|B\|} - \frac{f'(\|B\|) + f'_+(0)}{2} \right) \\
 &\leq f(A)A^{-2} - f(B)B^{-2} - f'_+(0)(A^{-1} - B^{-1}) \\
 &\leq 2\|B-A\| \|A^{-1}\|^2 \\
 &\quad \times \left(\|A^{-1}\| f(\|A^{-1}\|^{-1}) - \frac{f'(\|A^{-1}\|^{-1}) + f'_+(0)}{2} \right).
 \end{aligned}$$

REMARK 2. Consider the operator convex function $f(t) = -\ln(t+1)$, $t \geq 0$. Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B-A \leq M$ for some constants α, δ, m, M . Then by (2.21) we derive

$$\begin{aligned}
 0 &\leq \frac{2m}{\delta^2} \left(\frac{\delta+2}{2(\delta+1)} - \frac{\ln(\delta+1)}{\delta} \right) \\
 &\leq B^{-2} \ln(B+1) - A^{-2} \ln(A+1) + A^{-1} - B^{-1} \\
 &\leq \frac{2M}{\alpha^2} \left(\frac{\alpha+2}{2(\alpha+1)} - \frac{\ln(\alpha+1)}{\alpha} \right).
 \end{aligned}$$

If $A, B > 0$ and $B-A > 0$, then by (2.22)

$$\begin{aligned}
 0 &\leq \frac{2}{\|B\|^2 \|(B-A)^{-1}\|} \left(\frac{\|B\|+2}{2(\|B\|+1)} - \frac{\ln(\|B\|+1)}{\|B\|} \right) \\
 &\leq B^{-2} \ln(B+1) - A^{-2} \ln(A+1) + A^{-1} - B^{-1} \\
 &\leq 2\|B-A\| \|A^{-1}\|^2 \left(\frac{1+2\|A^{-1}\|}{2(\|A^{-1}\|+1)} - \|A^{-1}\| \ln(\|A^{-1}\|^{-1}+1) \right).
 \end{aligned}$$

3. More Examples

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then

$$\mathcal{D}(e_{-a})(t) := \int_0^\infty \frac{\exp(-a\lambda)}{t + \lambda} d\lambda = E_1(at) \exp(at), \quad t \geq 0,$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du, \quad t \geq 0.$$

For $a = 1$ we have

$$\mathcal{D}(e_{-1})(t) := \int_0^\infty \frac{\exp(-\lambda)}{t + \lambda} d\lambda = E_1(t) \exp(t), \quad t \geq 0.$$

Since $E_1'(t) = -\frac{e^{-t}}{t}$, $t > 0$, then

$$\mathcal{D}'(e_{-a})(t) = E_1'(at) \exp(at) + E_1(at) (\exp(at))' = aE_1(at) \exp(at) - \frac{1}{t}.$$

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α , δ , m , M . Then by (2.6) we get

$$\begin{aligned} 0 &\leq m \left[\frac{1}{\delta} - aE_1(a\delta) \exp(a\delta) \right] \\ &\leq E_1(aA) \exp(aA) - E_1(aB) \exp(aB) \\ &\leq M \left[\frac{1}{\alpha} - aE_1(a\alpha) \exp(a\alpha) \right], \end{aligned}$$

for $a > 1$, and in particular

$$\begin{aligned} 0 &\leq m \left[\frac{1}{\delta} - E_1(\delta) \exp(\delta) \right] \\ &\leq E_1(A) \exp(A) - E_1(B) \exp(B) \\ &\leq M \left[\frac{1}{\alpha} - E_1(\alpha) \exp(\alpha) \right]. \end{aligned}$$

If $A, B > 0$ and $B - A > 0$, then by (2.16),

$$\begin{aligned} 0 &\leq \left\| (B - A)^{-1} \right\|^{-1} \left[\|B\|^{-1} - aE_1(a\|B\|) \exp(a\|B\|) \right] \\ &\leq E_1(aA) \exp(aA) - E_1(aB) \exp(aB) \\ &\leq \|B - A\| \left[\|A^{-1}\| - aE_1\left(a\|A^{-1}\|^{-1}\right) \exp\left(a\|A^{-1}\|^{-1}\right) \right], \end{aligned}$$

for $a > 1$, and in particular

$$\begin{aligned} 0 &\leq \left\| (B - A)^{-1} \right\|^{-1} \left[\|B\|^{-1} - E_1(\|B\|) \exp(\|B\|) \right] \\ &\leq E_1(A) \exp(A) - E_1(B) \exp(B) \\ &\leq \|B - A\| \left[\|A^{-1}\| - E_1\left(\|A^{-1}\|^{-1}\right) \exp\left(\|A^{-1}\|^{-1}\right) \right]. \end{aligned}$$

More examples of such transforms are

$$\mathcal{D}(w_{1/(\ell^2+a^2)})(t) := \int_0^\infty \frac{1}{(t+\lambda)(\lambda^2+a^2)} d\lambda = \frac{\pi t - 2a \ln(t/a)}{2a(t^2+a^2)}, \quad t \geq 0$$

and

$$\mathcal{D}(w_{\ell/(\ell^2+a^2)})(t) := \int_0^\infty \frac{\lambda}{(t+\lambda)(\lambda^2+a^2)} d\lambda = \frac{\pi a + 2t \ln(t/a)}{2a(t^2+a^2)}, \quad t \geq 0$$

for $a > 0$.

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [3], [4], [5], [8] and [9].

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MATHEMATICS, COLLEGE OF ENGINEERING & SCIENCE
VICTORIA UNIVERSITY
PO Box 14428
MELBOURNE CITY
MC 800
AUSTRALIA
AND
DST-NRF CENTRE OF EXCELLENCE
IN THE MATHEMATICAL AND STATISTICAL SCIENCES
SCHOOL OF COMPUTER SCIENCE & APPLIED MATHEMATICS
UNIVERSITY OF THE WITWATERSRAND
JOHANNESBURG
SOUTH AFRICA
e-mail: sever.dragomir@vu.edu.au
URL: <http://rgmia.org/dragomir>