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FUZZY WEAK FILTERS OF SHEFFER STROKE HILBERT ALGEBRAS

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Abstract. The (fuzzy) filter of the Sheffer stroke Hilbert algebra was addressed by Oner, Katican and Borumand Saeid. The weak version of the filter was discussed by Jun and Oner. In this manuscipt, with the fuzzy version of the weak filter in mind, the notions of fuzzy weak filters and $(\in, \in\vee q)$-fuzzy weak filters are introduced, and their properties are explored. Conditions under which t-level set, Q_t-set and t-$\in\vee q$-set become weak filters are explored in relation to fuzzy weak filters and $(\in, \in\vee q)$-fuzzy weak filters. The relationship and characterization of the fuzzy weak filter and the $(\in, \in\vee q)$-fuzzy weak filter are investigated.

1. Introduction

The Sheffer operation "|" (the so-called Sheffer stroke in [1]) was first introduced by Sheffer [12] in 1913. In propositional calculus and Boolean functions, the Sheffer stroke represents a logical operation that is equivalent to the negation of the conjunction operation, and is expressed in ordinary language as "not both". The Sheffer stroke of P and Q is the negation of their conjunction, that is, $P|Q \Leftrightarrow \neg(P\wedge Q)$. By De Morgan's laws, it is also equivalent to the disjunction of the negations of P and Q, that is, $P|Q \Leftrightarrow \neg P\vee\neg Q$. The Sheffer stroke has been applied to several algebraic structures, for example, Boolean algebra,

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MV-algebra, BL-algebra, BCK-algebra, BE-algebra, ortholattices, and Hilbert algebra, etc. (see [2, 4, 5, 6, 9, 10]). Oner et al. [7] discussed the (fuzzy) filters of a Sheffer stroke Hilbert algebra. Also, Oner et al. [8] discussed deductive systems in Sheffer stroke Hilbert algebras. Jun and Oner [3] discussed weak filters and multipliers in Sheffer stroke Hilbert algebras. They first introduced the concept of weak filters that have weakened the filter conditions in the Schaefer stroke Hilbert algebra and investigated several properties.

The purpose of this paper is to address fuzzy versions of weak filters in the Shaper stroke Hilbert algebra. We introduce the notion of fuzzy weak filters and investigate the relevant properties. We discuss the characterization of fuzzy weak filters, and consider the conditions under which the fuzzy set becomes a fuzzy weak filter. We create a set $H_0 := \{x \in H \mid \xi(x) \neq 0\}$ with respect to a fuzzy set ξ and explore the environment in which it can be a weak filter. We provide conditions for the t-level set and Q_t-set of ξ to be weak filters of $\mathcal{H} := (H, |)$. We consider a chracterization of a fuzzy weak filter based on level set. Also, we introduce the notion of $(\in, \in\vee q)$-fuzzy weak filters based on the fuzzy point, and investigate the relevant properties. We look at the relationship between fuzzy weak filters and $(\in, \in\vee q)$-fuzzy weak filters, and find conditions under which $(\in, \in\vee q)$-fuzzy weak filters become fuzzy weak filters. We explore the conditions under which fuzzy sets become $(\in, \in\vee q)$-fuzzy weak filters. We investigate the situation regarding Q_t-sets and t-$\in\vee q$-sets in relation to $(\in, \in\vee q)$-fuzzy weak filters.

2. Preliminaries

Definition 2.1 ([12]). Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation "$|$" is said to be *Sheffer stroke* or *Sheffer operation* if it satisfies:

(s1) $(\forall a, b \in A)\ (a|b = b|a)$,

(s2) $(\forall a, b \in A)\ ((a|a)|(a|b) = a)$,

(s3) $(\forall a, b, c \in A)\ (a|((b|c)|(b|c)) = ((a|b)|(a|b))|c)$,

(s4) $(\forall a, b, c \in A)\ ((a|((a|a)|(b|b)))|(a|((a|a)|(b|b))) = a)$.

Definition 2.2 ([8]). A *Sheffer stroke Hilbert algebra* is a groupoid $\mathcal{H} := (H, |)$ with a Sheffer stroke "$|$" that satisfies:

(sH1) $(x|((A)|(A)))|(((B)|((C)|(C)))|((B)|((C)|(C)))) = x|(x|x)$,
where $A := y|(z|z)$, $B := x|(y|y)$ and $C := x|(z|z)$,

(sH2) $x|(y|y) = y|(x|x) = x|(x|x) \ \Rightarrow\ x = y$

for all $x, y, z \in H$.

Let $\mathcal{H} := (H, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation "$\preceq$" on H is defined as follows:

$$(\forall a, b \in H)(a \preceq b \;\Leftrightarrow\; a|(b|b) = 1).$$

We observe that the relation "$\preceq$" is a partial order in a Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ (see [8]). Recall that the Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ satisfies the identity $a|(a|a) = b|(b|b)$, which is denoted by 1, for all $a, b \in H$ (see [8]).

Proposition 2.3 ([8]). *Every Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ satisfies:*

$$(\forall a \in H)(a|(a|a) = 1),$$

$$(\forall a \in H)(a|(1|1) = 1), \tag{2.1}$$

$$(\forall a \in H)(1|(a|a) = a),$$

$$(\forall a, b \in H)(a \preceq b|(a|a)),$$

$$(\forall a, b \in H)((a|(b|b))|(b|b) = (b|(a|a))|(a|a)),$$

$$(\forall a, b \in H)\left(((a|(b|b))|(b|b))|(b|b) = a|(b|b)\right),$$

$$(\forall a, b, c \in H)\left(a|((b|(c|c))|(b|(c|c))) = b|((a|(c|c))|(a|(c|c)))\right),$$

$$(\forall a, b, c \in H)(a \preceq b \;\Rightarrow\; c|(a|a) \preceq c|(b|b),\ b|(c|c) \preceq a|(c|c)),$$

$$(\forall a, b, c \in H)(a|((b|(c|c))|(b|(c|c))) = (a|(b|b))|((a|(c|c))|(a|(c|c)))).$$

By (2.1), we know that the element 1 is the greatest element in $\mathcal{H} := (H, |)$ with respect to the order $\preceq$.

Definition 2.4 ([7]). Let $\mathcal{H} := (H, |)$ be a Sheffer stroke Hilbert algebra. A subset F of H is called a *filter* of $\mathcal{H} := (H, |)$ if it satisfies:

$$1 \in F, \tag{2.2}$$

$$(\forall a, b \in H)(b \in F \;\Rightarrow\; a|(b|b) \in F), \tag{2.3}$$

$$(\forall a, b, c \in H)(b, c \in F \;\Rightarrow\; (a|(b|c))|(b|c) \in F).$$

Let $\mathcal{H} := (H, |)$ be a Sheffer stroke Hilbert algebra. If a subset F of H satisfies (2.2) and (2.3), we say that F is a *weak filter* of $\mathcal{H} := (H, |)$ (see [3]).

A fuzzy set ξ in a set H of the form

$$\xi(b) := \begin{cases} t \in (0,1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\langle a/t \rangle$.

For a fuzzy set ξ in a set H, we say that a fuzzy point $\langle a/t \rangle$ is

(i) *contained* in ξ, denoted by $\langle a/t \rangle \in \xi$, (see [11]) if $\xi(a) \geq t$,

(ii) *quasi-coincident* with ξ, denoted by $\langle a/t \rangle \, q \, \xi$, (see [11]) if $\xi(a) + t > 1$.

If a fuzzy point $\langle a/t \rangle$ is contained in ξ or is quasi-coincident with ξ, we denote it $\langle a/t \rangle \in\!\vee q \, \xi$. If $\langle a/t \rangle \, \alpha \, \xi$ is not established for $\alpha \in \{\in, q, \in\!\vee q\}$, it is denoted by $\langle a/t \rangle \, \overline{\alpha} \, \xi$.

Given $t \in (0,1]$ and a fuzzy set ξ in a set H, consider the following sets

$$(\xi, t)_{\in} := \{x \in H \mid \langle x/t \rangle \in \xi\} \quad \text{and} \quad (\xi, t)_q := \{x \in H \mid \langle x/t \rangle \, q \, \xi\}$$

which are called a *t-level set* and a *Q_t-set* of ξ, respectively, in H. Also, we consider the set

$$(\xi, t)_{\in \vee q} := \{x \in H \mid \langle x/t \rangle \in\!\vee q \, \xi\}$$

which is called the *t-$\in\!\vee q$-set* of ξ. It is clear that $(\xi, t)_{\in \vee q} = (\xi, t)_{\in} \cup (\xi, t)_q$.

Definition 2.5 ([7]). Let $\mathcal{H} := (H, |)$ be a Sheffer stroke Hilbert algebra. A fuzzy set ξ in H is called a *fuzzy filter* of $\mathcal{H} := (H, |)$ if it satisfies:

$$(\forall a \in H)(\xi(1) \geq \xi(a)), \tag{2.4}$$

$$(\forall a, b \in H)(\xi(a|(b|b)) \geq \xi(b)), \tag{2.5}$$

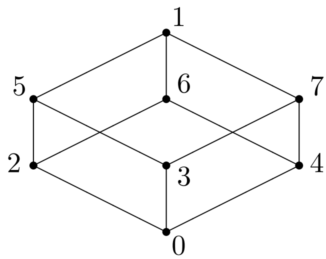
$$(\forall a, b, c \in H)(\xi((a|(b|c))|(b|c)) \geq \min\{\xi(b), \xi(c)\}).$$

3. Fuzzy weak filters

This section deals with the so-called fuzzy weak filter, which has weakened the conditions of the fuzzy filter. In what follows, $\mathcal{H} := (H, |)$ stands for a Sheffer stroke Hilbert algebra, unless otherwise stated.

Definition 3.1. A fuzzy set ξ in H is called a *fuzzy weak filter* of $\mathcal{H} := (H, |)$ if it satisfies (2.4) and (2.5).

Example 3.2. Let $H = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with the following Hasse diagram:



Define a Sheffer stroke "|" on H by Table 1

Table 1. Cayley table for the Sheffer stroke "|"

|	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	0

Then $\mathcal{H} := (H, |)$ is a Sheffer stroke Hilbert algebra (see [8]). We define a fuzzy set ξ in H as follows:

$$\xi\colon H \to [0,1], \quad x \mapsto \begin{cases} 0.69 & \text{if } x = 1, \\ 0.55 & \text{if } x \in \{5, 6\}, \\ 0.47 & \text{if } x = 7, \\ 0.38 & \text{otherwise.} \end{cases}$$

It is routine to verify that ξ is a fuzzy weak filter of $\mathcal{H} := (H, |)$.

It is clear that every fuzzy filter is a fuzzy weak filter, but the converse may not be true. In fact, consider the Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ in Example 3.2 and let ξ be a fuzzy set in H given by

$$\xi\colon H \to [0,1], \quad x \mapsto \begin{cases} 0.77 & \text{if } x = 1, \\ 0.68 & \text{if } x \in \{6, 7\}, \\ 0.53 & \text{otherwise.} \end{cases}$$

It is routine to verify that ξ is a fuzzy weak filter of $\mathcal{H} := (H, |)$. But it is not a fuzzy filter of $\mathcal{H} := (H, |)$ since

$$\xi((4|(6|7))|(6|7)) = \xi(4) = 0.53 \not\geq 0.68 = \min\{\xi(6), \xi(7)\}.$$

THEOREM 3.3. *For every nonempty subset F of H, let ξ_F be a fuzzy set in H defined as follows:*

$$\xi_F \colon H \to [0,1], \quad x \mapsto \begin{cases} t & \text{if } x \in F, \\ s & \text{otherwise} \end{cases}$$

where $s, t \in [0,1]$ with $s < t$. Then ξ_F is a fuzzy weak filter of $\mathcal{H} := (H, |)$ if and only if F is a weak filter of $\mathcal{H} := (H, |)$.

PROOF. Assume that ξ_F is a fuzzy weak filter of $\mathcal{H} := (H, |)$. Clearly, $1 \in F$ since $\xi_F(1) = t$. Let $x, y \in H$ be such that $y \in F$. Then $\xi_F(x|(y|y)) \geq \xi_F(y) = t$, and so $\xi_F(x|(y|y)) = t$. Thus $x|(y|y) \in F$, and therefore F is a weak filter of $\mathcal{H} := (H, |)$.

Conversely, suppose that F is a weak filter of $\mathcal{H} := (H, |)$. Then $\xi_F(1) = t \geq \xi_F(x)$ for all $x \in H$ since $1 \in F$. Let $x, y \in H$. If $y \notin F$, then $\xi_F(y) = s \leq \xi_F(x|(y|y))$. If $y \in F$, then $x|(y|y) \in F$, and thus $\xi_F(x|(y|y) = t = \xi_F(y)$. Hence ξ_F is a fuzzy weak filter of $\mathcal{H} := (H, |)$. □

COROLLARY 3.4. *For every $a \in H$ and $s, t \in [0,1]$ with $s < t$, the fuzzy set ξ_a in H defined by*

$$\xi_a \colon H \to [0,1], \quad x \mapsto \begin{cases} t & \text{if } x \in \vec{a}, \\ s & \text{otherwise} \end{cases}$$

is a fuzzy weak filter of $\mathcal{H} := (H, |)$ where $\vec{a} = \{x \in H \mid a|(x|x) = 1\}$.

PROOF. Since $\vec{a}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $a \in H$ (see [3]), it follows from Theorem 3.3 that ξ_a is a fuzzy weak filter of $\mathcal{H} := (H, |)$ for all $a \in H$. □

COROLLARY 3.5. *For every $a \in H$ and $s, t \in [0,1]$ with $s < t$, the fuzzy set ξ^a in H defined by*

$$\xi^a \colon H \to [0,1], \quad x \mapsto \begin{cases} t & \text{if } x \in H^a, \\ s & \text{otherwise} \end{cases}$$

is a fuzzy weak filter of $\mathcal{H} := (H, |)$ where $H^a = \{x \in H \mid a|(x|x) = x\}$.

PROOF. Note that H^a is a weak filter of $\mathcal{H} := (H, |)$ for all $a \in H$ (see [3]). Hence ξ^a is a fuzzy weak filter of $\mathcal{H} := (H, |)$ for all $a \in H$ by Theorem 3.3. □

COROLLARY 3.6. *Let F be a subset of H. For every $a \in H$ and $s, t \in [0, 1]$ with $s < t$, define a fuzzy set ξ_F^a in H as follows:*

$$\xi_F^a \colon H \to [0,1], \quad x \mapsto \begin{cases} t & \text{if } x \in F_a, \\ s & \text{otherwise} \end{cases}$$

where $F_a = \{z \in H \mid z = a|(x|x),\, x \in F\}$. If F is a weak filter of $\mathcal{H} := (H, |)$, then ξ_F^a is a fuzzy weak filter of $\mathcal{H} := (H, |)$.

PROOF. If F is a weak filter of $\mathcal{H} := (H, |)$, then F_a is a weak filter of $\mathcal{H} := (H, |)$ (see [3]). Thus ξ_F^a is a fuzzy weak filter of $\mathcal{H} := (H, |)$ by Theorem 3.3. □

COROLLARY 3.7. *Let F be a subset of H. For every $s, t \in [0, 1]$ with $s < t$, define a fuzzy set ξ_F^* in H as follows:*

$$\xi_F^* \colon H \to [0,1], \quad x \mapsto \begin{cases} t & \text{if } x \in F^*, \\ s & \text{otherwise} \end{cases}$$

where $F^* := \{x \in H \mid (\forall y \in F)(x|(y|y) = 1 \;\Rightarrow\; y = 1)\}$. If F is a weak filter of $\mathcal{H} := (H, |)$, then ξ_F^* is a fuzzy weak filter of $\mathcal{H} := (H, |)$.

PROOF. If F is a weak filter of $\mathcal{H} := (H, |)$, then F^* is a weak filter of $\mathcal{H} := (H, |)$ (see [3]). Thus ξ_F^* is a fuzzy weak filter of $\mathcal{H} := (H, |)$ by Theorem 3.3. □

Given a fuzzy set ξ in H, we consider the following set:

$$H_0 := \{x \in H \mid \xi(x) \neq 0\}.$$

It is clear that if ξ is a nonzero fuzzy set in H, that is, $\xi(a) \neq 0$ for some $a \in H$, then $H_0 \neq \emptyset$.

We explore conditions for the set H_0 to be a weak filter.

THEOREM 3.8. *If ξ is a nonzero fuzzy weak filter of $\mathcal{H} := (H, |)$, then the set H_0 is a weak filter of $\mathcal{H} := (H, |)$.*

PROOF. Let ξ be a nonzero fuzzy weak filter of $\mathcal{H} := (H, |)$. If $1 \notin H_0$, then $\xi(1) = 0$ and so $\xi(x) = 0$ for all $x \in H$. Hence $H_0 = \emptyset$, a contradiction. Hence $1 \in H_0$. If $y \in H_0$, then $\xi(x|(y|y)) \geq \xi(y) \neq 0$ for all $x \in H$ by (2.5) and so $x|(y|y) \in H_0$ for all $x \in H$. Thus H_0 is a weak filter of $\mathcal{H} := (H, |)$. □

THEOREM 3.9. *If a nonzero fuzzy set ξ in H satisfies:*

$$(\forall x \in H)(\forall t \in (0,1])(\langle x/t\rangle \in \xi \;\Rightarrow\; \langle 1/t\rangle \, q\, \xi), \tag{3.1}$$

$$(\forall x, y \in H)(\forall t \in (0,1])(\langle y/t\rangle \in \xi \;\Rightarrow\; \langle x|(y|y)/t\rangle \, q\, \xi), \tag{3.2}$$

then the set H_0 is a weak filter of $\mathcal{H} := (H, |)$.

PROOF. Since $\langle x/\xi(x)\rangle \in \xi$ for all $x \in H$, we have $\langle 1/\xi(x)\rangle \, q\, \xi$ by (3.1). If $1 \notin H_0$, then $\xi(1) = 0$ and thus $\langle 1/\xi(x)\rangle \, \overline{q}\, \xi$. This is a contradiction, and so $1 \in H_0$. Let $y \in H_0$. Then $\xi(y) \neq 0$. Note that $\langle y/\xi(y)\rangle \in \xi$. Hence $\langle x|(y|y)/\xi(y)\rangle \, q\, \xi$ for all $x \in H$ by (3.2), and so $\xi(x|(y|y)) + \xi(y) > 1$. If $x|(y|y) \notin H_0$, then $\xi(x|(y|y)) = 0$ and thus $\xi(y) > 1$, a contradiction. Hence $x|(y|y) \in H_0$, and therefore H_0 is a weak filter of $\mathcal{H} := (H, |)$. □

THEOREM 3.10. *If a nonzero fuzzy set ξ in H satisfies:*

$$(\forall x \in H)(\forall t \in (0,1])(\langle x/t\rangle \, q\, \xi \;\Rightarrow\; \langle 1/t\rangle \in \xi), \tag{3.3}$$

$$(\forall x, y \in H)(\forall t \in (0,1])(\langle y/t\rangle \, q\, \xi \;\Rightarrow\; \langle x|(y|y)/t\rangle \in \xi), \tag{3.4}$$

then the set H_0 is a weak filter of $\mathcal{H} := (H, |)$.

PROOF. Since $H_0 \neq \emptyset$, there exists $a \in H_0$ and so $\xi(a) \neq 0$. Thus $\xi(a)+1 > 1$, i.e., $\langle a/1\rangle \, q\, \xi$ which implies from (3.3) that $\langle 1/1\rangle \in \xi$. Thus $1 \in H_0$. Let $y \in H_0$. Then $\xi(y) \neq 0$, and so $\xi(y) + 1 > 1$, i.e., $\langle y/1\rangle \, q\, \xi$. Using (3.4), we have $\langle x|(y|y)/1\rangle \in \xi$. Hence $\xi(x|(y|y)) = 1 \neq 0$, i.e., $x|(y|y) \in H_0$. Therefore H_0 is a weak filter of $\mathcal{H} := (H, |)$. □

THEOREM 3.11. *If a nonzero fuzzy set ξ in H satisfies:*

$$(\forall x \in H)(\forall t \in (0,1])(\langle x/t\rangle \, q\, \xi \;\Rightarrow\; \langle 1/t\rangle \, q\, \xi), \tag{3.5}$$

$$(\forall x, y \in H)(\forall t \in (0,1])(\langle y/t\rangle \, q\, \xi \;\Rightarrow\; \langle x|(y|y)/t\rangle \, q\, \xi), \tag{3.6}$$

then the set H_0 is a weak filter of $\mathcal{H} := (H, |)$.

PROOF. Since $H_0 \neq \emptyset$, there exists $a \in H_0$ and so $\xi(a) \neq 0$. Thus $\xi(a)+1 > 1$, i.e., $\langle a/1\rangle \, q\, \xi$ which implies from (3.5) that $\langle 1/1\rangle \, q\, \xi$. Thus $\xi(1)+1 > 1$, and so $\xi(1) \neq 0$. Hence $1 \in H_0$. If $y \in H_0$, then $\xi(y) \neq 0$, and so $\xi(y) + 1 > 1$, i.e., $\langle y/1\rangle \, q\, \xi$. It follows from (3.6) that $\langle x|(y|y)/1\rangle \, q\, \xi$. Hence $\xi(x|(y|y)) + 1 > 1$, and so $\xi(x|(y|y)) \neq 0$ which shows that $x|(y|y) \in H_0$. Therefore H_0 is a weak filter of $\mathcal{H} := (H, |)$. □

We provide conditions for the t-level set and Q_t-set of ξ to be weak filters of $\mathcal{H} := (H, |)$.

Theorem 3.12. *If a fuzzy set* ξ *in* H *satisfies:*

$$(\forall x \in H)(\xi(x) \leq \max\{\xi(1), 0.5\}), \tag{3.7}$$

$$(\forall x, y \in H)(\xi(y) \leq \max\{\xi(x|(y|y)), 0.5\}), \tag{3.8}$$

then the nonempty t-level set $(\xi, t)_\in$ *of* ξ *is a weak filter of* $\mathcal{H} := (H, |)$ *for all* $t \in (0.5, 1]$.

Proof. Let $t \in (0.5, 1]$ be such that $(\xi, t)_\in \neq \emptyset$. Then there exists $a \in (\xi, t)_\in$, and so $\max\{\xi(1), 0.5\} \geq \xi(a) \geq t > 0.5$ by (3.7). Hence $\xi(1) \geq t$, and so $1 \in (\xi, t)_\in$. Let $x, y \in H$ be such that $y \in (\xi, t)_\in$. Then $\xi(y) \geq t > 0.5$ and so $\max\{\xi(x|(y|y)), 0.5\} \geq \xi(y) \geq t > 0.5$ by (3.8). Hence $\xi(x|(y|y)) \geq t$ and thus $x|(y|y) \in (\xi, t)_\in$. Therefore $(\xi, t)_\in$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0.5, 1]$. □

Theorem 3.13. *The converse of Theorem* 3.12 *is also true, that is, if the nonempty t-level set* $(\xi, t)_\in$ *of* ξ *is a weak filter of* $\mathcal{H} := (H, |)$ *for all* $t \in (0.5, 1]$, *then* ξ *satisfies* (3.7) *and* (3.8).

Proof. Assume that the nonempty t-level set $(\xi, t)_\in$ of ξ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0.5, 1]$. If the condition (3.7) is not establish, then

$$(\exists a \in H)(\xi(a) > \max\{\xi(1), 0.5\}).$$

Hence $t := \xi(a) \in (0.5, 1]$ and $a \in (\xi, t)_\in$. But $1 \notin (\xi, t)_\in$, a contradiction. Thus $\xi(x) \leq \max\{\xi(1), 0.5\}$ for all $x \in H$. Suppose that (3.8) is not valid. Then $\xi(b) > \max\{\xi(a|(b|b)), 0.5\}$ for some $a, b \in H$. If we take $s := \xi(b)$, then $s \in (0.5, 1]$ and $b \in (\xi, s)_\in$. But $\max\{\xi(a|(b|b)), 0.5\} < s$ induces $a|(b|b) \notin (\xi, s)_\in$, which is a contradiction. Therefore $\xi(y) \leq \max\{\xi(x|(y|y)), 0.5\}$ for all $x, y \in H$. □

Theorem 3.14. *If* ξ *is a fuzzy weak filter of* $\mathcal{H} := (H, |)$, *then its nonempty* Q_t*-set* $(\xi, t)_q$ *is a weak filter of* $\mathcal{H} := (H, |)$ *for all* $t \in (0, 1]$.

Proof. Let $t \in (0, 1]$ be such that $(\xi, t)_q \neq \emptyset$. Since $\xi(1) \geq \xi(x)$ for $x \in (\xi, t)_q$, we have $\xi(1) \geq \xi(x) > 1 - t$ and so $\langle 1/t \rangle \, q \, \xi$, i.e., $1 \in (\xi, t)_q$. If $y \in (\xi, t)_q$, then $\xi(y) + t > 1$ and thus $\xi(x|(y|y)) \geq \xi(y) > 1 - t$. Hence $\langle x|(y|y)/t \rangle \, q \, \xi$, i.e., $x|(y|y) \in (\xi, t)_q$ for all $x \in H$. Therefore $(\xi, t)_q$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 1]$. □

Proposition 3.15. *Given a fuzzy set* ξ *in* H*, if its* Q_t*-set* $(\xi, t)_q$ *is a weak filter of* $\mathcal{H} := (H, |)$ *for all* $t \leq 0.5$*, then the following assertions are established:*

$$(\forall t \in (0, 0.5])(1 \in (\xi, t)_{\in}), \tag{3.9}$$

$$(\forall x, y \in H)(\forall t \in (0, 0.5])\left(\langle y/t\rangle \, q \, \xi \;\Rightarrow\; x|(y|y) \in (\xi, t)_{\in}\right). \tag{3.10}$$

Proof. Since $1 \in (\xi, t)_q$ and $t \leq 0.5$, we have $\xi(1) > 1 - t \geq t$ and so $1 \in (\xi, t)_{\in}$. Let $x, y \in H$ and $t \in (0, 0.5]$ be such that $\langle y/t\rangle \, q \, \xi$. Then $y \in (\xi, t)_q$. It follows from (2.3) that $x|(y|y) \in (\xi, t)_q$. Hence $\xi(x|(y|y)) > 1 - t \geq t$, and thus $x|(y|y) \in (\xi, t)_{\in}$. □

Proposition 3.16. *Given a fuzzy set* ξ *in* H*, if its* Q_t*-set* $(\xi, t)_q$ *is a weak filter of* $\mathcal{H} := (H, |)$ *for all* $t \in (0.5, 1]$*, then* ξ *satisfies:*

$$(\forall x, y \in H)(\forall t \in (0.5, 1])\left(\langle y/t\rangle \in \xi \;\Rightarrow\; x|(y|y) \in (\xi, t)_q\right). \tag{3.11}$$

Proof. Let $x, y \in H$ and $t \in (0.5, 1]$ be such that $\langle y/t\rangle \in \xi$. Then $\xi(y) \geq t > 1 - t$, i.e., $\langle y/t\rangle \, q \, \xi$, and so $y \in (\xi, t)_q$. Since $(\xi, t)_q$ is a weak filter of $\mathcal{H} := (H, |)$, we have $x|(y|y) \in (\xi, t)_q$. □

Corollary 3.17. *Every fuzzy weak filter* ξ *of* $\mathcal{H} := (H, |)$ *satisfies* (3.9)*,* (3.10) *and* (3.11)*.*

Theorem 3.18. *If a fuzzy set* ξ *in* H *satisfies:*

$$(\forall x \in H)(\forall t \in (0.5, 1])(\langle x/t\rangle \, q \, \xi \;\Rightarrow \langle 1/t\rangle \in\!\vee q \, \xi), \tag{3.12}$$

$$(\forall x, y \in H)(\forall t \in (0.5, 1])(\langle y/t\rangle \, q \, \xi \;\Rightarrow \langle x|(y|y)/t\rangle \in\!\vee q \, \xi), \tag{3.13}$$

then the nonempty Q_t*-set* $(\xi, t)_q$*,* $t \in (0.5, 1]$*, of* ξ *is a weak filter of* $\mathcal{H} := (H, |)$*.*

Proof. Let $t \in (0.5, 1]$ be such that $(\xi, t)_q \neq \emptyset$. Then there exists $x \in (\xi, t)_q$, and so $\langle x/t\rangle \, q \, \xi$. Hence $\langle 1/t\rangle \in \vee q \, \xi$ by (3.12), that is, $\langle 1/t\rangle \in \xi$ or $\langle 1/t\rangle \, q \, \xi$. If $\langle 1/t\rangle \, q \, \xi$, then $1 \in (\xi, t)_q$. If $\langle 1/t\rangle \in \xi$, then $\xi(1) \geq t > 1 - t$, i.e., $\langle 1/t\rangle \, q \, \xi$ and so $1 \in (\xi, t)_q$. Let $y \in (\xi, t)_q$. Then $\langle y/t\rangle \, q \, \xi$, which implies from (3.13) that $\langle x|(y|y)/t\rangle \in\!\vee q \, \xi$, that is, $\langle x|(y|y)/t\rangle \in \xi$ or $\langle x|(y|y)/t\rangle \, q \, \xi$ for all $x \in H$. If $\langle x|(y|y)/t\rangle \, q \, \xi$, then $x|(y|y) \in (\xi, t)_q$ for all $x \in H$. If $\langle x|(y|y)/t\rangle \in \xi$, then $\xi(x|(y|y)) \geq t > 1 - t$, i.e., $\langle x|(y|y)/t\rangle \, q \, \xi$. Hence $x|(y|y) \in (\xi, t)_q$ for all $x \in H$. Therefore $(\xi, t)_q$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0.5, 1]$. □

Proposition 3.19. *Let F be a weak filter of $\mathcal{H} := (H, |)$. If a fuzzy set ξ in H satisfies $\xi(x) = 0$ for $x \in H \setminus F$ and $x \in (\xi, 0.5)_\in$ for $x \in F$, then it satisfies:*

$$(3.14) \qquad (\forall x \in H)(\forall t \in (0,1])(\langle x/t\rangle \, q \, \xi \;\Rightarrow \langle 1/t\rangle \in\vee q \, \xi),$$

$$(3.15) \qquad (\forall x, y \in H)(\forall t \in (0,1])(\langle y/t\rangle \, q \, \xi \;\Rightarrow \langle x|(y|y)/t\rangle \in\vee q \, \xi).$$

Proof. Let $x \in H$ and $t \in (0,1]$ be such that $\langle x/t\rangle \, q \, \xi$. Then $\xi(x) + t > 1$. If $x \in H \setminus F$, then $\xi(x) = 0$ and thus $t > 1$. This is impossible, and so $x \in F$. Hence $\xi(x) \geq 0.5$. If $\langle 1/t\rangle \, \overline{\in} \, \xi$, then $t > \xi(1) \geq 0.5$ and so $\xi(1) + t > 1$, i.e., $\langle 1/t\rangle \, q \, \xi$. Thus (3.14) is valid. Let $x, y \in H$ and $t \in (0,1]$ be such that $\langle y/t\rangle \, q \, \xi$. Then $\xi(y) + t > 1$, and so $y \in F$. Since F is a weak filter of $\mathcal{H} := (H, |)$, we get $x|(y|y) \in F$ and thus $x|(y|y) \in (\xi, 0.5)_\in$, i.e., $\xi(x|(y|y)) \geq 0.5$. If $\langle x|(y|y)/t\rangle \, \overline{q} \, \xi$, then $1 \geq \xi(x|(y|y)) + t \geq 0.5 + t$ and hence $t \leq 0.5 \leq \xi(x|(y|y))$. Thus $\langle x|(y|y)/t\rangle \in \xi$ which shows that (3.15) is valid. □

Corollary 3.20. *Let F be a weak filter of $\mathcal{H} := (H, |)$. If a fuzzy set ξ in H satisfies $\xi(x) = 0$ for $x \in H \setminus F$ and $x \in (\xi, 0.5)_\in$ for $x \in F$, then the Q_t-set $(\xi, t)_q$ of ξ is a weak filter of $\mathcal{H} := (H, |)$.*

Now we consider a chracterization of a fuzzy weak filter based on level set.

Theorem 3.21. *Given a fuzzy set ξ in H, the following are equivalent.*

(i) *ξ is a fuzzy weak filter of $\mathcal{H} := (H, |)$.*
(ii) *The nonempty t-level set $(\xi, t)_\in$ of ξ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in [0,1]$.*

Proof. Assume that ξ is a fuzzy weak filter of $\mathcal{H} := (H, |)$ and let $t \in [0,1]$ be such that $(\xi, t)_\in \neq \emptyset$. Then $\xi(1) \geq \xi(x) \geq t$ for some $x \in (\xi, t)_\in$, and so $1 \in (\xi, t)_\in$. Let $y \in (\xi, t)_\in$. Then $\xi(y) \geq t$, and so $\xi(x|(y|y)) \geq \xi(y) \geq t$ for all $x \in H$ by (2.5). Hence $x|(y|y) \in (\xi, t)_\in$, which shows that $(\xi, t)_\in$ is a weak filter of $\mathcal{H} := (H, |)$.

Conversely, suppose that the nonempty t-level set $(\xi, t)_\in$ of ξ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in [0,1]$. If $\xi(1) < \xi(a)$ for some $a \in H$, then $a \in (\xi, \xi(a))_\in$ but $1 \notin (\xi, \xi(a))_\in$, a contradiction. Thus $\xi(1) \geq \xi(x)$ for all $x \in H$. Suppose that $\xi(b) > \xi(a|(b|b))$ for some $a, b \in H$. If we take $t := \xi(b)$, then $b \in (\xi, t)_\in$ and $a|(b|b) \notin (\xi, t)_\in$. This is a contradiction, and thus $\xi(x|(y|y)) \geq \xi(y)$ for all $x, y \in H$. Consequently, ξ is a fuzzy weak filter of $\mathcal{H} := (H, |)$. □

4. $(\in, \in\vee q)$-fuzzy weak filters

In this section, we introduce the concept of $(\in, \in \vee q)$-fuzzy weak filters and discuss its properties.

Definition 4.1. A fuzzy set ξ in H is called an $(\in, \in\vee q)$-*fuzzy weak filter* of $\mathcal{H} := (H, |)$ if it satisfies:

$$(\forall x \in H)(\forall t \in (0,1])(\langle x/t\rangle \in \xi \;\Rightarrow\; \langle 1/t\rangle \in\vee q\,\xi), \tag{4.1}$$

$$(\forall x, y \in H)(\forall t \in (0,1])(\langle y/t\rangle \in \xi \;\Rightarrow\; \langle x|(y|y)/t\rangle \in\vee q\,\xi). \tag{4.2}$$

Example 4.2. Consider the Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ in Example 3.2. We define a fuzzy set ξ in H as follows:

$$\xi\colon H \to [0,1], \quad x \mapsto \begin{cases} 0.77 & \text{if } x = 1, \\ 0.83 & \text{if } x \in \{6,7\}, \\ 0.47 & \text{if } x = 5, \\ 0.35 & \text{otherwise.} \end{cases}$$

It is routine to verify that ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$.

It is clear that every fuzzy weak filter is an $(\in, \in \vee q)$-fuzzy weak filter. But the converse may not be true. In fact, the $(\in, \in \vee q)$-fuzzy weak filter ξ of $\mathcal{H} := (H, |)$ in Example 4.2 is not a fuzzy weak filter of $\mathcal{H} := (H, |)$ since $\xi(1) \ngeq \xi(6)$.

Theorem 4.3. *If an $(\in, \in\vee q)$-fuzzy weak filter ξ of $\mathcal{H} := (H, |)$ satisfies:*

$$(\forall x \in H)(\xi(x) \leq 0.5), \tag{4.3}$$

then it is a fuzzy weak filter of $\mathcal{H} := (H, |)$.

Proof. Let ξ be an $(\in, \in \vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$ that satisfies (4.3). Note that $\langle x/\xi(x)\rangle \in \xi$ for all $x \in H$. Hence $\langle 1/\xi(x)\rangle \in \vee q\,\xi$, that is, $\langle 1/\xi(x)\rangle \in \xi$ or $\langle 1/\xi(x)\rangle\, q\,\xi$ by (4.1). If $\langle 1/\xi(x)\rangle \in \xi$, then $\xi(1) \geq \xi(x)$. If $\langle 1/\xi(x)\rangle\, q\,\xi$, then $\xi(1) + \xi(x) > 1$ and so $\xi(1) > 1 - \xi(x) \geq 0.5 \geq \xi(x)$ by (4.3). Since $\langle y/\xi(y)\rangle \in \xi$ for all $y \in H$, it follows from (4.2) that $\langle x|(y|y)/\xi(y)\rangle \in\vee q\,\xi$, i.e., $\langle x|(y|y)/\xi(y)\rangle \in \xi$ or $\langle x|(y|y)/\xi(y)\rangle\, q\,\xi$ for all $x \in H$. If $\langle x|(y|y)/\xi(y)\rangle \in \xi$, then $\xi(x|(y|y)) \geq \xi(y)$. If $\langle x|(y|y)/\xi(y)\rangle\, q\,\xi$, then $\xi(x|(y|y)) + \xi(y) > 1$ and so $\xi(x|(y|y)) > 1 - \xi(y) \geq 0.5 \geq \xi(y)$ by (4.3). Therefore ξ is a fuzzy weak filter of $\mathcal{H} := (H, |)$. □

Corollary 4.4. *If an* $(\in, \in \vee q)$*-fuzzy weak filter* ξ *of* $\mathcal{H} := (H, |)$ *satisfies* (4.3)*, then the nonempty t-level set* $(\xi, t)_{\in}$ *of* ξ *is a weak filter of* $\mathcal{H} := (H, |)$.

We explore the conditions in which a fuzzy set becomes an $(\in, \in\!\vee q)$-fuzzy weak filter.

Theorem 4.5. *If a fuzzy set* ξ *in* H *satisfies* (3.14) *and* (3.15)*, then* ξ *is an* $(\in, \in\!\vee q)$*-fuzzy weak filter of* $\mathcal{H} := (H, |)$.

Proof. Let ξ be a fuzzy set in H that satisfies (3.14) and (3.15). Let $x \in H$ and $t \in (0, 1]$ be such that $\langle x/t\rangle \in \xi$. Then $\xi(x) \geq t$. If $\langle 1/t\rangle \,\overline{\in\vee q}\, \xi$, then $\langle 1/t\rangle \,\overline{\in}\, \xi$ and $\langle 1/t\rangle \,\overline{q}\, \xi$. Hence $\xi(1) < t$ and $\xi(1) + t \leq 1$, which imply that $\xi(1) < 0.5$. Thus $\xi(1) < \min\{t, 0.5\}$, and so

$$1 - \xi(1) > 1 - \min\{t, 0.5\} = \max\{1 - t, 0.5\} \geq \max\{1 - \xi(x), 0.5\}.$$

It follows that $\xi(x) + s > 1$, that is, $\langle x/s\rangle \, q\, \xi$ for $s := 1 - \xi(1)$. Using (3.14) induces $\langle 1/s\rangle \in \vee q\, \xi$, i.e., $\langle 1/s\rangle \in \xi$ or $\langle 1/s\rangle \, q\, \xi$. This is a contradiction, and therefore $\langle 1/t\rangle \in\!\vee q\, \xi$. Let $y \in H$ and $t \in (0, 1]$ be such that $\langle y/t\rangle \in \xi$. Then $\xi(y) \geq t$. Assume that $\langle x|(y|y)/t\rangle \,\overline{\in\vee q}\, \xi$ for some $x \in H$. Then $\langle x|(y|y)/t\rangle \,\overline{\in}\, \xi$, i.e., $\xi(x|(y|y)) < t$ and $\langle x|(y|y)/t\rangle \,\overline{q}\, \xi$, i.e., $\xi(x|(y|y)) + t \leq 1$. Hence $\xi(x|(y|y)) < \min\{t, 0.5\}$, and so there exists $\delta \in (0, 1]$ such that

$$\begin{aligned} 1 - \xi(x|(y|y)) &\geq \delta > 1 - \min\{t, 0.5\} \\ &= \max\{1 - t, 0.5\} \\ &\geq \max\{1 - \xi(y), 0.5\}. \end{aligned}$$

Thus $\langle y/\delta\rangle \, q\, \xi$, which implies from (3.15) that $\langle x|(y|y)/\delta\rangle \in \vee q\, \xi$, i.e., $\langle x|(y|y)/\delta\rangle \in \xi$ or $\langle x|(y|y)/\delta\rangle \, q\, \xi$ for all $x \in H$. If $\langle x|(y|y)/\delta\rangle \in \xi$, then

$$1 - \xi(x|(y|y)) \leq 1 - \delta < 1 - 0.5 = 0.5 < \delta,$$

a contradiction. If $\langle x|(y|y)/\delta\rangle \, q\, \xi$, then $\xi(x|(y|y)) > 1 - \delta$ which is a contradiction. Hene $\langle x|(y|y)/t\rangle \in\!\vee q\, \xi$ for all $x \in H$. Therefore ξ is an $(\in, \in\!\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$. □

Theorem 4.6. *Assume that every fuzzy point has the value t in* $(0, 0.5]$. *Then every* $(\in, \in \vee q)$*-fuzzy weak filter* ξ *of* $\mathcal{H} := (H, |)$ *satisfies* (3.14) *and* (3.15).

Proof. Let ξ be an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$. Let $x \in H$ and $t \in (0, 1]$ be such that $\langle x/t\rangle \, q \, \xi$. Then $\xi(x) > 1 - t \geq t$ since every fuzzy point has the value t in $(0, 0.5]$, and thus $\langle x/t\rangle \in \xi$. Hence $\langle 1/t\rangle \in\vee q \, \xi$ by (4.1). Let $x, y \in H$ and $t \in (0, 1]$ be such that $\langle y/t\rangle \, q \, \xi$. Then $\xi(y) > 1 - t \geq t$, i.e., $\langle y/t\rangle \in \xi$. It follows from (4.2) that $\langle x|(y|y)/t\rangle \in\vee q \, \xi$. □

Theorem 4.7. *If a fuzzy set ξ in H satisfies:*

$$(\forall x \in H)(\forall t \in (0, 1])(\langle x/t\rangle \in\vee q \, \xi \;\Rightarrow \langle 1/t\rangle \in\vee q \, \xi), \tag{4.4}$$

$$(\forall x, y \in H)(\forall t \in (0, 1])(\langle y/t\rangle \in\vee q \, \xi \;\Rightarrow \langle x|(y|y)/t\rangle \in\vee q \, \xi), \tag{4.5}$$

then ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$.

Proof. It is straightforward because if $\langle x/t\rangle \in \xi$, then $\langle x/t\rangle \in\vee q \, \xi$ for all $x \in H$ and $t \in (0, 1]$. □

Theorem 4.8. *A fuzzy set ξ in H is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$ if and only if it satisfies:*

$$(\forall x \in H)(\xi(1) \geq \min\{\xi(x), 0.5\}), \tag{4.6}$$

$$(\forall x, y \in H)(\xi(x|(y|y)) \geq \min\{\xi(y), 0.5\}). \tag{4.7}$$

Proof. Assume that ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$. For every $x \in H$, we consider two cases: $\xi(x) \geq 0.5$ and $\xi(x) < 0.5$. If $\xi(x) \geq 0.5$, then $\langle x/0.5\rangle \in \xi$ and so $\langle 1/0.5\rangle \in\vee q \, \xi$ by (4.1). It follows that $\xi(1) \geq 0.5$, i.e., $\langle 1/0.5\rangle \in \xi$. Otherwise, $\xi(1) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Suppose that $\xi(x) < 0.5$. If $\xi(1) < \xi(x)$, then $\xi(1) < t \leq \xi(x)$ for some $t \in (0, 0.5)$. Thus $\langle x/t\rangle \in \xi$ and $\langle 1/t\rangle \,\overline{\in}\, \xi$. Since $\xi(1) + t < 1$, we get $\langle 1/t\rangle \, \overline{q} \, \xi$ and hence $\langle 1/t\rangle \, \overline{\in\vee q} \, \xi$. This is a contradiction, and therefore $\xi(1) \geq \xi(x)$. Consequently, $\xi(1) \geq \min\{\xi(x), 0.5\}$ for all $x \in H$. Let $x, y \in H$. If $\xi(y) < 0.5$, then $\xi(x|(y|y)) \geq \xi(y)$. In fact, if not then $\xi(x|(y|y)) < t \leq \xi(y)$ for some $t \in (0, 0.5)$. Thus $\langle y/t\rangle \in \xi$ but $\langle x|(y|y)/t\rangle \, \overline{\in\vee q} \, \xi$ which is a contradiction. If $\xi(y) \geq 0.5$, then $\langle y/0.5\rangle \in \xi$ and so $\langle x|(y|y)/0.5\rangle \in\vee q \, \xi$ by (4.2). Hence $\xi(x|(y|y)) \geq 0.5$ because if $\xi(x|(y|y)) < 0.5$, then $\xi(x|(y|y)) + 0.5 < 0.5 + 0.5 = 1$, i.e., $\langle x|(y|y)/0.5\rangle \, \overline{q} \, \xi$ which is a contradiction. Therefore $\xi(x|(y|y)) \geq \min\{\xi(y), 0.5\}$ for all $x, y \in H$.

Conversely, suppose that ξ satisfies (4.6) and (4.7). Let $x \in H$ and $t \in (0, 1]$ be such that $\langle x/t\rangle \in \xi$. Then $\xi(x) \geq t$. Suppose that $\xi(1) < t$, i.e., $\langle 1/t\rangle \,\overline{\in}\, \xi$. If $\xi(x) < 0.5$, then $\xi(1) \geq \min\{\xi(x), 0.5\} = \xi(x) \geq t$, a contradiction. Hence $\xi(x) \geq 0.5$, and so $\xi(1) + t > 2\xi(1) \geq 2\min\{\xi(x), 0.5\} = 1$. Hence $\langle 1/t\rangle \in\vee q \, \xi$. Let $x, y \in H$ and $t \in (0, 1]$ be such that $\langle y/t\rangle \in \xi$. Assume that $\langle x|(y|y)/t\rangle \,\overline{\in}\, \xi$. Then $\xi(x|(y|y)) \geq 0.5$ because if not, then $\xi(x|(y|y)) \geq$

$\min\{\xi(y), 0.5\} = \xi(y) \geq t$, a contradiction. Thus $\xi(x|(y|y)) + t > 2\xi(x|(y|y)) \geq 2\min\{\xi(y), 0.5\} = 1$, that is, $\langle x|(y|y)/t\rangle \, q \, \xi$ and so $\langle x|(y|y)/t\rangle \in\!\vee q \, \xi$. Consequently, ξ is an $(\in, \in\!\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$. □

THEOREM 4.9. *A fuzzy set ξ in H is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$ if and only if the nonempty t-level set $(\xi, t)_\in$ of ξ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 0.5]$.*

PROOF. Assume that ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$ and let $t \in (0, 0.5]$ be such that $(\xi, t)_\in \neq \emptyset$. Then there exists $a \in (\xi, t)_\in$ and so $\xi(a) \geq t$, that is, $\langle a/t\rangle \in \xi$. Hence $\langle 1/t\rangle \in\vee q \, \xi$ by (4.1), and so $\xi(1) \geq t$ or $\xi(1) + t > 1$. If $\xi(1) \geq t$, then $1 \in (\xi, t)_\in$. If $\xi(1) + t > 1$, then $\xi(1) > 1 - t \geq t$ since $t \leq 0.5$. Hence $1 \in (\xi, t)_\in$. Let $x \in H$ and $y \in (\xi, t)_\in$. Then $\xi(y) \geq t$, that is, $\langle y/t\rangle \in \xi$. It follows from (4.2) that $\langle x|(y|y)/t\rangle \in\!\vee q \, \xi$. Hence $\xi(x|(y|y)) \geq t$ or $\xi(x|(y|y)) + t > 1$. If $\xi(x|(y|y)) \geq t$, then $x|(y|y) \in (\xi, t)_\in$. If $\xi(x|(y|y)) + t > 1$, then $\xi(x|(y|y)) > 1 - t \geq t$ since $t \leq 0.5$. Hence $x|(y|y) \in (\xi, t)_\in$. Therefore $(\xi, t)_\in$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 0.5]$.

Conversely, suppose that $(\xi, t)_\in$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 0.5]$. Suppose that there exists $a \in H$ such that $\xi(1) < \min\{\xi(a), 0.5\}$. Taking $t := \min\{\xi(a), 0.5\}$ induces $\langle a/t\rangle \in \xi$ but $\langle 1/t\rangle \,\overline{\in}\, \xi$, i.e., $1 \notin (\xi, t)_\in$. This is a contradiction, and thus $\xi(1) \geq \min\{\xi(x), 0.5\}$ for all $x \in H$. Assume that $\xi(a|(b|b)) < \min\{\xi(b), 0.5\}$ for some $a, b \in H$. Then $\langle b/\min\{\xi(b), 0.5\}\rangle \in \xi$, i.e., $b \in (\xi, \min\{\xi(b), 0.5\})_\in$ but $\langle a|(b|b)/\min\{\xi(b), 0.5\}\rangle \,\overline{\in}\, \xi$, i.e., $a|(b|b) \notin (\xi, \min\{\xi(b), 0.5\})_\in$. This is a contradiction, and so $\xi(x|(y|y)) \geq \min\{\xi(y), 0.5\}$ for all $x, y \in H$. It follows from Theorem 4.8 that ξ is an $(\in, \in\!\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$. □

THEOREM 4.10. *If ξ is an $(\in, \in\!\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$, then the nonempty Q_t-set $(\xi, t)_q$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0.5, 1]$.*

PROOF. Assume that ξ is an $(\in, \in\!\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$ and let $t \in (0.5, 1]$ be such that $(\xi, t)_q \neq \emptyset$. Let $x \in (\xi, t)_q$. Then $\langle x/t\rangle \, q \, \xi$ and so $\xi(x) + t > 1$. It follows from (4.6) in Theorem 4.8 that

$$\xi(1) + t \geq \min\{\xi(x), 0.5\} + t = \min\{\xi(x) + t, 0.5 + t\} > 1.$$

Hence $\langle 1/t\rangle \, q \, \xi$ and so $1 \in (\xi, t)_q$. Now, let $x \in H$ and $y \in (\xi, t)_q$. Then $\langle y/t\rangle \, q \, \xi$ and so $\xi(y) + t > 1$. Using (4.7) in Theorem 4.8, we have

$$\xi(x|(y|y)) + t \geq \min\{\xi(y), 0.5\} + t = \min\{\xi(y) + t, 0.5 + t\} > 1,$$

i.e., $\langle x|(y|y)/t\rangle \, q \, \xi$. Thus $x|(y|y) \in (\xi, t)_q$. Therefore $(\xi, t)_q$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0.5, 1]$. □

THEOREM 4.11. *If ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$, then the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 0.5]$.*

PROOF. Assume that ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$ and let $t \in (0, 0.5]$ be such that $(\xi, t)_{\in\vee q} \neq \emptyset$. Let $x \in (\xi, t)_{\in\vee q}$. Then $\langle x/t\rangle \in \vee q\, \xi$, that is, $\langle x/t\rangle \in \xi$ or $\langle x/t\rangle\, q\, \xi$. If $\langle x/t\rangle \in \xi$, then $\langle 1/t\rangle \in\vee q\, \xi$ by (4.1). If $\langle x/t\rangle\, q\, \xi$, then $\xi(x) > 1 - t \geq t$ since $t \leq 0.5$. Hence $\langle x/t\rangle \in \xi$ which implies from (4.1) that $\langle 1/t\rangle \in\vee q\, \xi$. Thus $1 \in (\xi, t)_{\in\vee q}$. Let $x \in H$ and let $y \in (\xi, t)_{\in\vee q}$. Then $\langle y/t\rangle \in\vee q\, \xi$, that is, $\langle y/t\rangle \in \xi$ or $\langle y/t\rangle\, q\, \xi$. If $\langle y/t\rangle \in \xi$, then $\langle x|(y|y)/t\rangle \in\vee q\, \xi$ by (4.2). If $\langle y/t\rangle\, q\, \xi$, then $\xi(y) > 1 - t \geq t$ since $t \leq 0.5$. It follows from (4.2) that $\langle x|(y|y)/t\rangle \in\vee q\, \xi$. Hence $x|(y|y) \in (\xi, t)_{\in\vee q}$, and therefore $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 0.5]$. □

COROLLARY 4.12. *If a fuzzy set ξ in H satisfies* (3.14) *and* (3.15)*, then the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 0.5]$.*

COROLLARY 4.13. *If a fuzzy set ξ in H satisfies* (4.4) *and* (4.5)*, then the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 0.5]$.*

THEOREM 4.14. *Given a fuzzy set ξ in H, if the t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 1]$, then ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$.*

PROOF. Let ξ be a fuzzy set in H such that $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 1]$. Assume that there exists $a \in H$ such that $\xi(1) < \min\{\xi(a), 0.5\}$. Then $\langle a/\min\{\xi(a), 0.5\}\rangle \in \xi$, and so

$$a \in (\xi, \min\{\xi(a), 0.5\})_{\in} \subseteq (\xi, \min\{\xi(a), 0.5\})_{\in\vee q}.$$

But $\langle 1/\min\{\xi(a), 0.5\}\rangle \,\overline{\in}\, \xi$ and

$$\xi(1) + \min\{\xi(a), 0.5\} < 2\min\{\xi(a), 0.5\} \leq 1,$$

i.e., $\langle 1/\min\{\xi(a), 0.5\}\rangle\, \overline{q}\, \xi$. Hence $\langle 1/\min\{\xi(a), 0.5\}\rangle\, \overline{\in\vee q}\, \xi$, and thus

$$1 \notin (\xi, \min\{\xi(a), 0.5\})_{\in\vee q}.$$

This is a contradiction, and hence $\xi(1) \geq \min\{\xi(x), 0.5\}$ for all $x \in H$. Suppose that $\xi(a|(b|b)) < \min\{\xi(b), 0.5\}$ for some $a, b \in H$. Then there exists $t \in (0, 0.5]$ such that

$$\xi(a|(b|b)) < t \leq \min\{\xi(b), 0.5\}.$$

It follows that $b \in (\xi, t)_{\in} \subseteq (\xi, t)_{\in\vee q}$. But $\langle a|(b|b)/t\rangle \,\overline{\in}\, \xi$ and

$$\xi(a|(b|b)) + t < 2t \leq 1, \text{ i.e., } \langle a|(b|b)/t\rangle \,\overline{q}\, \xi.$$

Hence $\langle a|(b|b)/t\rangle \,\overline{\in\vee q}\, \xi$, i.e., $a|(b|b) \notin (\xi, t)_{\in\vee q}$ which is a contradiction. Thus $\xi(x|(y|y)) \geq \min\{\xi(y), 0.5\}$ for all $x, y \in H$. Using Theorem 4.8, we know that ξ is an $(\in, \in\vee q)$-fuzzy weak filter of $\mathcal{H} := (H, |)$. □

Theorem 4.15. *If a fuzzy set ξ in H satisfies* (3.14) *and* (3.15)*, then the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0.5, 1]$.*

Proof. Suppose that $(\xi, t)_{\in\vee q} \neq \emptyset$ for all $t \in (0.5, 1]$, and take $x \in (\xi, t)_{\in\vee q}$. Then $\langle x/t\rangle \in\vee q\, \xi$, and so $\langle x/t\rangle \in \xi$ or $\langle x/t\rangle\, q\, \xi$. If $\langle x/t\rangle\, q\, \xi$, then $\langle 1/t\rangle \in\vee q\, \xi$ by (3.14) and so $1 \in (\xi, t)_{\in\vee q}$. If $\langle x/t\rangle \in \xi$, then $\xi(x) + t \geq 2t > 1$, that is, $\langle x/t\rangle\, q\, \xi$ since $t \in (0.5, 1]$. It follows from (3.14) that $\langle 1/t\rangle \in\vee q\, \xi$, that is, $1 \in (\xi, t)_{\in\vee q}$. Let $x \in H$ and $y \in (\xi, t)_{\in\vee q}$. Then $\langle y/t\rangle \in\vee q\, \xi$, and so $\langle y/t\rangle \in \xi$ or $\langle y/t\rangle\, q\, \xi$. If $\langle y/t\rangle \in \xi$, then $\xi(y) + t \geq 2t > 1$, i.e., $\langle y/t\rangle\, q\, \xi$ since $t \in (0.5, 1]$. It follows from (3.15) that $\langle x|(y|y)/t\rangle \in\vee q\, \xi$. Hence $x|(y|y) \in (\xi, t)_{\in\vee q}$. If $\langle y/t\rangle\, q\, \xi$, then $\langle x|(y|y)/t\rangle \in\vee q\, \xi$ by (3.15), and thus $x|(y|y) \in (\xi, t)_{\in\vee q}$. Therefore $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0.5, 1]$. □

When Corollary 4.12 and Theorem 4.15 are combined, the following corollary is obtained.

Corollary 4.16. *If a fuzzy set ξ in H satisfies* (3.14) *and* (3.15)*, then the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 1]$.*

Theorem 4.17. *Given a fuzzy set ξ in H, if the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 1]$, then ξ satisfies* (3.14) *and* (3.15)*.*

Proof. Assume that the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 1]$. Let $x \in H$ and $t \in (0, 1]$ be such that $\langle x/t\rangle\, q\, \xi$. Then $x \in (\xi, t)_q \subseteq (\xi, t)_{\in\vee q}$, i.e., $(\xi, t)_{\in\vee q} \neq \emptyset$. Hence $1 \in (\xi, t)_{\in\vee q}$, which shows that $\langle 1/t\rangle \in\vee q\, \xi$. So (3.14) is valid. Let $x, y \in H$ and $t \in (0, 1]$ be such that $\langle y/t\rangle\, q\, \xi$. Then $y \in (\xi, t)_q \subseteq (\xi, t)_{\in\vee q}$ which implies from (2.3) that $x|(y|y) \in (\xi, t)_{\in\vee q}$. Thus $\langle x|(y|y)/t\rangle \in\vee q\, \xi$, which proves (3.15). □

Using Corollary 4.16 and Theorem 4.17, we have the following corollary.

Corollary 4.18. *A fuzzy set ξ satisfies* (3.14) *and* (3.15) *if and only if the nonempty t-$\in\vee q$-set $(\xi, t)_{\in\vee q}$ is a weak filter of $\mathcal{H} := (H, |)$ for all $t \in (0, 1]$.*

5. Conclusion

In Boolean functions and propositional calculus, the Sheffer stroke denotes a logical operation that is equivalent to the negation of the conjunction operation, expressed in ordinary language as “not both”. The Sheffer stroke is a powerful logical connective that can be used to simplify and express complex logical statements. It is often used in digital electronics to design logic circuits. Everyone can find more information about the Sheffer stroke at

https://en.wikipedia.org/wiki/Sheffer_stroke.

Oner, Katican and Borumand Saeid applied the Sheffer stroke to a logical algebra called the Hilbert algebra, and introduced the Sheffer stroke Hilbert algebra. This will provide various research topics for logical algebra researchers. The purpose of this paper is to study fuzzy versions of weak filters, which are weaker than filters, in Shaper stroke Hilbert algebra. We introduced the notions of fuzzy weak filters and $(\in, \in \vee q)$-fuzzy weak filters, and explored their properties. We discussed conditions under which t-level set, Q_t-set and t-$\in\vee q$-set become weak filters in relation to fuzzy weak filters and $(\in, \in\vee q)$-fuzzy weak filters. We studied the relationship and characterization of the fuzzy weak filter and the $(\in, \in\vee q)$-fuzzy weak filter. Several theorems in this paper are tightly linked to a specific value of 0.5 in the membership function. By replacing this value of 0.5 with an arbitrary value between 0 and 1, we can generalize the results in this paper. And this is a follow-up work that we will pursue in the future.

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