FINDING INCOMES OF THE HM-NETWORK WITH ONE-TYPE MESSAGES BYPASS OF SYSTEMS

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Abstract. The object of research is an open HM-network with one-type messages bypass of systems in the transient behavior. Two cases are considered: when incomes from transitions between network states are deterministic functions depending on states and time, and network systems are single-line, and when incomes from transitions between network states are functions depending on random variables. The purpose of the research is to find the expected incomes of such a network in both cases on condition that the probabilities of messages bypasses of systems network and the parameters of incoming flow of messages and services depend on time. Examples are considered.

General information

In [1], a study was conducted in open exponential networks with multiline queuing systems (QS) with one-type messages bypass of systems in transient behavior. In [2] we found the nonstationary state probabilities and the average characteristics of the network in the case, when the probabilities of messages bypasses of systems network and parameters of incoming flow of messages and services depend on time.

In this paper there are considered open HM (Howard-Matalytski) - network incomes of such type, when the probabilities of messages bypasses of systems network depend on time. Here we will consider two cases: when incomes from transitions between network states are deterministic functions depending on states and time and the QS networks are single-line, and when incomes from transitions between network states are functions depending on the random variables (RV).

1. Formulation of the problem

Consider an open exponential QN with one-type messages, consisting of n QS $S_1, S_2, ..., S_n$. Messages during the transition from one to another QS bring some income and, accordingly, the income of the first system is reduced by that amount. It is necessary to find the expected incomes systems in the network during the time t o condition that we know its state at the initial time t_0.

Let m_i be the number of identical service lines in the QS S_i, I_i - a vector of dimension n, consisting of zeros except the i-th component, which is equal to 1, $i=\overline{1,n}$; p_{ij} - the transition probability of the message after service in the system S_i into the system S_j, $i,j=\overline{0,n}$. We assume that the system S_0 is the external environment. Let us consider the case when the parameters of the incoming flow of messages and services depend on time, i.e. the time interval $[t,t-\Delta t)$ in the network receives a message with a probability $\lambda(t)\Delta t+o(\Delta t)$, and if at the time t of service on the line i-th QS located in a message, at the range $[t,t+\Delta t)$ of its services will end with a probability $\mu_i(t)\Delta t+o(\Delta t)$, $i=\overline{1,n}$. The message is sent to the i-th QS with probability p_{0i}, $\sum_{i=1}^{n} p_{0i}=1$. The message sent to this QS from the external environment at a moment of time t, with a probability $f^{(i)}(k,t)$ when the network is in a state (k,t), joins the queue, and the probability $1-f^{(i)}(k,t)$ is not attached to the queue, regardless of the handled (i.e., its time of service with a probability of 1 is equal to zero). If the message has been served in the i-th QS, it is likely to be sent immediately to the j-th QS with probability p_{ij}, and leaves the QN with the probability p_{i0}, $\sum_{j=0}^{n} p_{ij}=1$, $i=1,...,n$.

Let $k(t)=(k,t)=(k_1,k_2,...,k_n,t)$ be the state vector of the network, where k_i - the number of messages at the moment t in the system S_i, $i=\overline{1,n}$; $\varphi_i(k,t)$ - the conditional probability that the message is delivered to the i-th QS at time t, when the network is in a state (k,t), will not be serviced by any of the QS; $\psi_{ij}(k,t)$ - the conditional probability that the message is delivered to the i-th QS outside at time t, when the network is in state (k,t), the first time, a service in j-th QS; $\alpha_i(k,t)$ - the conditional probability that the message, served in the i-th queuing system at time t, when the network is in a state (k,t), will no longer be served in any of QS; $\beta_{ij}(k,t)$ - the conditional probability that the message, served in the i-th queuing system at time t, when the network is in state (k,t) for the first time then receives services in the j-th QS, $i,j=\overline{1,n}$.

2. Finding the expected incomes, when incomes from transitions between network states are deterministic functions, depend on states and time

Let $v_i(k,t)$ - total expected income, which the system S_i gets during time t, if at the initial moment the network is in state k, and it is assumed that this function is differentiable in t; $r_i(k)$ - income of system S_i at a time when the network is in state k; $r_{0i}(k+I_i,t)$ - income of system S_i, when the network makes a transition from state (k,t) in the state $(k+I_i,t+\Delta t)$ during time Δt; $-R_{i0}(k-I_i,t)$ - system income, if the network makes a transition from state (k,t) in the state $(k-I_i,t+\Delta t)$; $r_{ij}(k+I_i-I_j,t)$ - income of system S_i (consumption or loss of system S_j), when the network changes its state from (k,t) to $(k+I_i-I_j,t+\Delta t)$ during time Δt, $i,j=\overline{1,n}$.

Suppose that the network is in state (k,t). During an interval of time it can remain in state k or move to states $(k-I_i)$, $(k+I_i)$, $(k-I_i-I_j)$, $i,j=\overline{1,n}$. If the network is in state $(k,t+\Delta t)$, then the expected income of system S_i will be $r_i(k)\Delta t$ plus the expected income $v_i(k,t)$, that it will receive the remaining time t units. The probability of this event is $1-\sum_{i=1}^{n}(\lambda(t)p_{0i}(1-\varphi_i(k,t))+\mu_i(t)(1-\beta_{ii}(k,t)))\Delta t+o(\Delta t)$. If the network goes to the state $(k+I_i,t+\Delta t)$ with probability $\lambda(t)p_{0i}\psi_{ii}(k+I_i,t)\Delta t+$ $+o(\Delta t)$, then income of system S_i is $[r_{0i}(k+I_i,t)-v_i(k+I_i,t)]$, and if in state $(k-I_i,t+\Delta t)$ with probability $\mu_i(t)\alpha_i(k-I_i,t)u(k_i)\Delta t+o(\Delta t)$, then income of the system is $[-R_{i0}(k-I_i,t)\quad v_i(k-I_i,t)]$, $i=\overline{1,n}$. Similarly, if the network goes from (k,t) to state $(k+I_i-I_j,t+\Delta t)$ with probability $\mu_j(t)\beta_{ii}(k+I_i-I_j,t)u(k_j)\Delta t+o(\Delta t)$, it brings the system S_i income of $r_{ij}(k+I_i-I_j,t)$ plus the expected net income for the remaining time, if the initial state of the network was the state $(k-I_i-I_j)$. Described above are summarized in Table 1.

Then, using the total probability formula for the mean value of the expected income of the system S_i a system of difference-differential equations (DDE) can be obtained:

$$\frac{dv_i(k,t)}{dt}=-\sum_{i=1}^{n}\left[\lambda(t)p_{0i}(1-\varphi_i(k,t))-\mu_i(t)(1-\beta_{ii}(k,t))\right]v_i(k,t)-$$

$$+\sum_{j=1}^{n}\left[\lambda(t)p_{0j}\psi_{jj}(k+I_j,t)v_i(k+I_j,t)+\mu_j(t)\alpha_j(k-I_j,t)u(k_j)v_i(k-I_j,t)\right]+$$

$$
\begin{aligned}
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\Big[\mu_j(t)\beta_{ji}\big(k+I_i-I_j,t\big)u(k_j)v_i(k+I_i-I_j,t)+\\
&\quad+\mu_i(t)\beta_{ij}\big(k-I_i+I_j,t\big)u(k_i)v_i(k-I_i+I_j,t)\Big]+\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\Big[\mu_j(t)\beta_{ji}\big(k+I_i-I_j,t\big)u(k_j)r_{ij}(k+I_i-I_j,t)+\\
&\quad+\mu_i(t)\beta_{ij}\big(k-I_i-I_j,t\big)u(k_i)r_{ji}(k-I_i+I_j,t)\Big]+\\
&+\sum_{\substack{c,s=1\\ c,s\neq i}}\mu_s(t)\beta_{sc}\big(k+I_c-I_s,t\big)u(k_s)v_i(k+I_c-I_s,t)+\\
&+\lambda(t)p_{0i}\psi_{ij}\big(k+I_i,t\big)r_{0i}(k+I_i,t)-\mu_i(t)\alpha_i\big(k-I_i,t\big)u(k_i)R_{i0}\big(k-I_i,t\big)+r_i\big(k\big),
\end{aligned}
\tag{1}
$$

Table 1

Possible transitions between network's states, their probability and incomes of the system S_i

Possible transitions between network's states	The transition probabilities	Incomes of system S_i of transitions between network's states
$(k,t)\to(k,t+\Delta t)$	$1-\sum_{j=1}^{n}\big(\lambda(t)p_{0i}(1-\varphi_i(k,t))+ \mu_j(t)(1-\beta_{ji}(k,t))\big)\Delta t+o(\Delta t)$	$r_i(k)\Delta t+v_i(k,t)$
$(k,t)\to(k+I_j,t+\Delta t)$, $j\neq i$	$\lambda(t)p_{0j}\psi_{ij}\big(k+I_j,t\big)\Delta t+o(\Delta t)$	$r_i(k)\Delta t+v_i(k+I_j,t)$
$(k,t)\to(k-I_j,t+\Delta t)$, $j\neq i$	$\mu_j(t)\alpha_j\big(k-I_j,t\big)u(k_j)\Delta t+o(\Delta t)$	$r_i(k)\Delta t+v_i(k-I_j,t)$
$(k,t)\to (k+I_c-I_s,t+\Delta t)$, $c,s\neq i$	$\mu_s(t)\beta_{sc}\big(k+I_c-I_s,t\big)u(k_s)\Delta t+o(\Delta t)$	$r_i(k)\Delta t+v_i(k+I_c-I_s,t)$
$(k,t)\to(k+I_i,t+\Delta t)$	$\lambda(t)p_{0i}\psi_{ij}\big(k+I_j,t\big)\Delta t+o(\Delta t)$	$r_{0i}(k+I_i,t)+v_i(k+I_i,t)$
$(k,t)\to(k-I_i,t+\Delta t)$	$\mu_i(t)\alpha_i\big(k-I_i,t\big)u(k_i)\Delta t+o(\Delta t)$	$-R_{i0}\big(k-I_i,t\big)+v_i(k-I_i,t)$
$(k,t)\to (k+I_i-I_j,t+\Delta t)$, $j\neq i$	$\mu_j(t)\beta_{ji}\big(k+I_i-I_j,t\big)u(k_j)\Delta t+o(\Delta t)$	$r_{ij}(k+I_i-I_j,t)+ +v_i(k-I_i-I_j,t)$
$(k,t)\to (k-I_i+I_j,t+\Delta t)$, $j\neq i$	$\mu_i(t)\beta_{ij}\big(k-I_i+I_j,t\big)u(k_i)\Delta t+o(\Delta t)$	$-r_{ji}(k-I_i+I_j,t)+ +v_i(k-I_i+I_j,t)$

where $u(x)=\begin{cases}1, x>0\\ 0, x\le 0\end{cases}$ - the Heaviside function. The number of equations in this system is the number of states of the network. Relations for the conditional probabilities $\varphi_i(k,t)$, $\psi_{ij}(k,t)$, $\alpha_i(k,t), \beta_{ij}(k,t)$, $i,j=\overline{1,n}$ are given at [1].

For closed networks, the system of equations (1) can be reduced to the final of the system of linear inhomogeneous ordinary differential equations (ODE) with constant coefficients, which in the matrix form can be written as

$$\frac{dV_i(t)}{dt}=Q_i(t)+A(t)V_i(t), \tag{2}$$

where: $V_i^T(t)=(v_i(1,t),v_i(2,t),...,v_i(L,t))$ - unknown vector of the system income S_i, L - number of states of the network. The solution of (1) can be found by using the direct method, or the method of Laplace transforms. However, we should not forget that the number of states of a closed QN equals $L=C_{n+K-1}^{n-1}$, where: K - number of messages served by the network, and it is quite large for a relatively small n and K, i.e. the number of equations in (1) will also be large enough. Experience has shown that the method of Laplace transforms can make calculations for networks with relatively small state space $(L<300)$; the direct method can be carried out for networks of larger dimension than the method of Laplace transforms.

Example 1

Consider a closed network with the following parameters: $n=5$, $K=10$. Let intensity be $\lambda(t)=\lambda t$, $\mu_i(t)=\mu_i\left[\cos(\omega_i t)+1\right]$, $i=\overline{1,5}$. The transition probabilities between the QS network are $p_{12}=p_{34}=p_{54}=1$, $p_{42}=p_{45}=0.3$, $p_{31}=p_{31}=p_{43}=$ $=0.4$, $p_{35}=0.2$. Because the network is closed, it equals the number of states $L=C_{n+K-1}^{n-1}=1001$. Write down some of them (0,0,0,0,7), (0,0,0,1,6), (0,0,0,2,5), (0,0,0,3,4), (0,0,0,4,3), (0,0,0,5,2), (0,0,0,6,1), (0,0,0,7,0), (0,0,1,0,6), (0,0,1,1,5) etc. Rename them from 1 to 1001. Assume that the probabilities of the messages to join the queue at time t is given by $f^{(i)}(t)=1-e^{-it}$, $i=\overline{1,5}$. Conditional probabilities $\varphi_i(t)$, $\psi_{ij}(t)$, $\alpha_i(t)$ and $\beta_{ij}(t)$ are given at [1]:

$$
\begin{aligned}
&\varphi_i(k,t)=\left(1-f^{(i)}(k,t)\right)\left(p_{i0}+\sum_{j=1}^{n}p_{ij}\varphi_j(k,t)\right),\ i=\overline{1,n},\\
&\psi_{ij}(k,t)=f^{(i)}(k,t)\delta_{ij}+(1-f^{(i)}(k,t))\sum_{l=1}^{n}p_{il}\psi_{lj}(k,t),\ i,j=\overline{1,n}\\
&\alpha_i(k,t)=p_{i0}+\sum_{j=1}^{n}p_{ij}\varphi_j(k-l_i,t),\ i=\overline{1,n},\\
&\beta_{ij}(k,t)=\sum_{l=1}^{n}p_{il}\psi_{lj}(k-l_i,t),\ i,j=\overline{1,n},
\end{aligned}
\tag{3}
$$

where δ_{ij} - the Kronecker delta. At [2] it was shown that they are cumbersome even at $n=3$. As the incomes of $r_i(k)$ were taken are some integers, $i=\overline{1,5}$. As incomes $r_{0i}(l,t)$, $R_{i0}(l,t)$, $r_{ij}(l,t)$, $r_{ji}(l,t)$ were taken linear in t not depending on states k functions: $at+b$, where: a are some real numbers, and b are some integers, $i,j=\overline{1,5}$, $i\neq j$. The initial conditions $V_i(0)$ - vector, consists of zeros, $i=\overline{1,5}$.

The numerical solution of the DDE (1) is obtained by using the mathematical calculations *NDSolve* package *Mathematica*, which by default is calculated with the help of the Runge-Kutta method of fourth order. The graph of system income S_3, on condition that at the initial moment network is in a state (1, 2, 3, 4, 5) is shown at Figure 1.

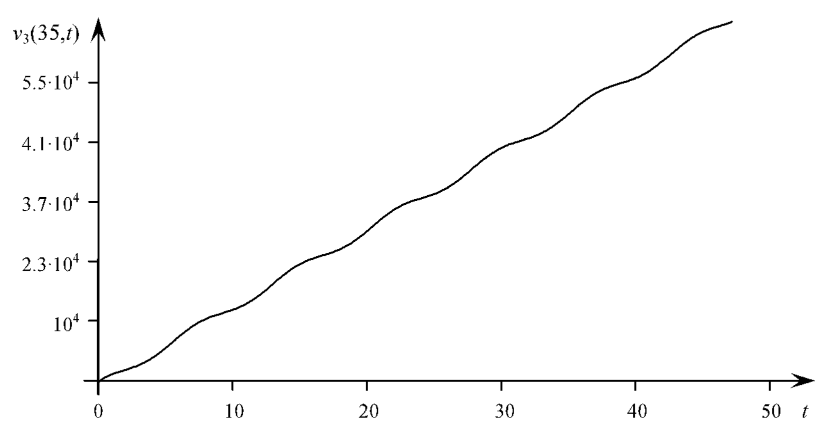

Fig. 1. The graph of income change of system S_3 at the time interval $[0;50]$

3. Finding the expected incomes, when incomes from transitions between network states are functions that depend on the random variables

Consider the case when the conditional probabilities $\varphi_i(k,t)$, $\psi_{ij}(k,t)$, $\alpha_i(k,t)$, $\beta_{ij}(k,t)$, do not depend on the state of the network, i.e. $\varphi_i(k,t)=\varphi_i(t)$, $\psi_{ij}(k,t)=\psi_{ij}(t)$, $\alpha_i(k,t)=\alpha_i(t)$, $\beta_{ij}(k,t)=\beta_{ij}(t)$, $i,j=\overline{1,n}$. As before, we consider the case when the revenue from the state transition networks are RV or functions that depend on them [3-5]. Let RV ξ_i - service time messages in the system S_i, distributed exponentially with the distribution function (DF) $F_{\xi_i}(t)=1-e^{-\mu_i(t)t}$, $i=\overline{1,n}$. Consider the dynamic of system S_i income Suppose that at the initial moment income of the system is equal to v_{i0}. We are interested in income of the system $V_i(t)$ at time moment t. We subdivide the interval $[0,t]$ into m equal parts of length $\Delta t=\dfrac{t}{m}$, considering the m large enough to find the income the system S_i write out the conditional probabilities of the events that may occur in the l-th time interval, $l=\overline{1,m}$. All possible transitions between network states, their probabilities and incomes of system S_i from such transitions are following:

1. With the probability $p_i^{(1)}(l,t,\Delta t)=\lambda(t)p_{0i}\psi_{ii}(t)\Delta t+o(\Delta t)$ the system S_i received message from the external environment that will bring it an in come of r_{0i}, where: r_{0i} - RV with the expectation (m.e.) $M(r_{0i})=a_{0i}$, $i=\overline{1,n}$.
2. With the probability $p_i^{(2)}(l,t,\Delta t)=\mu_i(t)\varepsilon_i^{(l)}\alpha_i(t)u(k_i^{(l)})\Delta t+o(\Delta t)$ message from the system S_i enters into the environment, with the income of the system S_i reduced by the value R_{i0}, where: R_{i0} - RV with m.e. $M(R_{i0})=b_{i0}$, $k_i^{(l)}$ - the number of messages in the system S_i (in a queue and serviced) on l-th time interval, $l=\overline{1,m}$, $i=\overline{1,n}$. Value $\varepsilon_i^{(l)}$ means the number of lines of messages for employment l-th time interval in the system S_i, $i=\overline{1,n}$.
3. The message from the system S_i transformed into the system S_j with the probability $p_{ij}^{(3)}(l,t,\Delta t)=\mu_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})\beta_{ij}(t)\Delta t+o(\Delta t)$, $j=\overline{1,n}$, $i\neq j$. Such a transition gives the system S_i income reduced by the value $R_{ij}(\xi_i)$, and the income of the system S_j increased by this value, in which $R_{ij}(\xi_i)$ - the casual income (loss) of the system S_i because of the transition from the message the system S_i

into system S_j, $M\{R_{ij}(\xi_i)\}=\int_0^\infty R_{ij}(t)dF_{\xi_i}(t)=\int_0^\infty R_{ij}(t)e^{-\mu_i(t)t}dt=\gamma_{ij}$, $i=\overline{1,n}$, $j=\overline{1,n}$, $i\neq j$.

4. With the probability

$$p^{(4)}(l,t,\Delta t)=1-\left(\lambda(t)\sum_{i=1}^{n}p_{0i}\sum_{\substack{j=1\\ j\neq i}}^{n}\psi_{ij}(t)-\sum_{i=1}^{n}\mu_i(t)\alpha_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})-\right.$$
$$+\sum_{i=1}^{n}\mu_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})\sum_{j=1}^{n}\beta_{ij}(t)-$$
$$\left.-\sum_{i=1}^{n}\left[\lambda(t)p_{0i}(1-\varphi_i(t))+\mu_i(t)(1-\beta_{ii}(t))\right]\right)\Delta t+o(\Delta t)$$

at the time interval Δt network status will not change.

In addition, for each small time interval Δt system S_i because of containing a message, increases the value of the income $\eta_i\Delta t$, where η_i - RV with m.e. $M\{\eta_i\}=\overline{\eta}_i$, $i=\overline{1,n}$. We also assume that RV r_{0i}, R_{0i}, $R_{ij}(\xi_i)$ are pairwise independent, $i=\overline{1,n}$, $j=\overline{1,n}$.

Let $\Delta V_{il}(\Delta t)$ - changes in the income of system S_i on l-th time interval associated with the transitions between the QS network applications. Then it follows from the above:

$$\Delta V_{il}(\Delta t)=\begin{cases} r_{0i}+\eta_i\Delta t \text{ with the probability } p_i^{(1)}(l,t,\Delta t),\\ -R_{i0}-\eta_i\Delta t \text{ with the probability } p_i^{(2)}(l,t,\Delta t),\\ -R_{ij}(\xi_i)-\eta_i\Delta t \text{ with the probability } p_{ij}^{(3)}(l,t,\Delta t),\ j=\overline{1,n},\ j\neq i,\\ \eta_i\Delta t \text{ with the probability } p^{(6)}(l,t,\Delta t). \end{cases} \quad (4)$$

The total income of the system S_i is equal to

$$V_i(t)=v_{i0}+\sum_{l=1}^{m}\Delta V_{il}(\Delta t)=v_{i0}+V_i(\Delta t), \quad (5)$$

where $V_i(t)=\sum_{l=1}^{m}\Delta V_{il}(\Delta t)$ - income of the system S_i from moving messages, $i=\overline{1,n}$.

We find an expression for the expected income of the system S_i at time t. Using the formula of total probability for the expectation, for a fixed realization of the process $k(t)$ we can write:

$$M\{\Delta V_{il}(\Delta t)/k(t)\}=(a_{0i}+\overline{\eta}_i\Delta t)p_i^{(1)}(l,t,\Delta t)+(-b_{i0}+\overline{\eta}_i\Delta t)p_i^{(2)}(l,t,\Delta t)+ \\ +(-\gamma_{ij}+\overline{\eta}_i\Delta t)p_{ij}^{(3)}(l,t,\Delta t)+\overline{\eta}_i p^{(4)}(l,t,\Delta t)\Delta t,\ i=\overline{1,n} \quad (6)$$

Further, substituting functions $p_i^{(1)}(l,t,\Delta t)$, $p_i^{(2)}(l,t,\Delta t)$, $p_{ij}^{(3)}(l,t,\Delta t)$ and $p^{(4)}(l,t,\Delta t)$ for suitable transition probabilities, we obtain

$$M\{\Delta V_{il}(\Delta t)/k(t)\}=(a_{0i}-\overline{\eta}_i\Delta t)\left(\lambda(t)\,p_{0i}\sum_{j=1}^{n}\psi_{ij}(t)\Delta t+o(\Delta t)\right)- \\ +(-b_{i0}+\overline{\eta}_i\Delta t)(\mu_i(t)\alpha_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})\Delta t+o(\Delta t))+ \\ +\left(-\sum_{j=1}^{n}\gamma_{ij}+\overline{\eta}_i\Delta t\right)\left(\mu_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})\sum_{j=1}^{n}\beta_{ij}(t)\Delta t+o(\Delta t)\right)+ \\ +\overline{\eta}_i\Delta t\left[1-\left(\lambda(t)\sum_{i=1}^{n}\sum_{i=1}^{r}p_{0i}\sum_{\substack{j=1\\ j\neq i}}^{n}\psi_{ij}(t)+\sum_{i=1}^{n}\mu_i(t)\alpha_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})+\sum_{i=1}^{n}\mu_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})\sum_{j=1}^{n}\beta_{ij}(t)-\right.\right. \\ \left.\left.-\sum_{i=1}^{n}\left[\lambda(t)p_{0i}(1-\varphi_i(t))+\mu_i(t)(1-\beta_{ii}(t))\right]\right)\Delta t-o(\Delta t)\right],\ i=\overline{1,n}. \quad (7)$$

Then, given that $m\Delta t=t$, expanding and making some transformations it can be written:

$$M\{V_i(t)/k(t)\}=\sum_{l=1}^{m}M\{\Delta V_{il}(\Delta t)/k(t)\}= \\ =\sum_{i=1}^{n-1}\left[a_{0i}\lambda(t)p_{0i}\sum_{j=1}^{n}\psi_{ij}(t)-\mu_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})\left(b_{i0}\alpha_i(t)+\sum_{j=1}^{n}\beta_{ij}(t)\gamma_{ij}\right)+\overline{\eta}_i\right]t- \\ -m\sum_{i=1}^{n-1}\overline{\eta}_i\left[\sum_{i=2}^{n}\left(\lambda(t)p_{0i}\sum_{j=1}^{n}\psi_{ij}(t)+\mu_i(t)\varepsilon_i^{(l)}u(k_i^{(l)})\left(\alpha_i(t)+\sum_{j=1}^{n}\beta_{ij}(t)\right)\right)-\right. \\ \left.-\sum_{i=1}^{n}\left[\lambda(t)p_{0i}(1-\varphi_i(t))+\mu_i(t)(1-\beta_{ii}(t))\right]\right]\Delta t+o(\Delta t),\ i=\overline{1,n} \quad (8)$$

Obtain expressions for the expected incomes $v_i(t)=M\{V_i(t)\}$. Averaging over $k(t)$ with the normalization condition $\sum_k P(k(t)=k)=1$, for the expected income of the system S_i we have

$$v_i(t) = M\{V_i(t)\} = \sum_k P(k(t)=k) M\{V_i(t)/k(t)\} =$$
$$= \left[a_{0i}\lambda(t)p_{0i}\sum_{j=1}^{n}\psi_{ij}(t) - \mu_i(t)\sum_{l=1}^{m}\varepsilon_i^{(l)}u(k_i^{(l)})\left(b_{i0}\alpha_i(t) + \sum_{j=1}^{n}\beta_{ij}(t)\gamma_{ij}\right) + \overline{\eta}_i \right] t -$$
$$-\sum_k P(k(t)=k)\overline{\eta}_i \left[\sum_{i=2}^{n}\left(\lambda(t)p_{0i}\sum_{j=1}^{n}\int_0^t \psi_{ij}(x)dx + \mu_i(t)\left(\alpha_i(t) + \sum_{j=1}^{n}\beta_{ij}(t)\right)\sum_{l=1}^{m}\varepsilon_i^{(l)}u(k_i^{(l)})\Delta t \right) - \right. \tag{9}$$
$$\left. -\sum_{i=1}^{n}\left[\lambda(t)p_{0i}(1-\varphi_i(t)) + \mu_i(t)(1-\beta_{ii}(t))\right]\right],\ i=\overline{1,n}.$$

We denote the expectation

$$M\{\varepsilon_i(t)u(k_i(t))\} = \rho_i(t),\ i=\overline{1,n}, \tag{10}$$

where $\rho_i(t)$ - the average number of employed service lines in the system S_i messages at the time moment t, $i=\overline{1,n}$.

At $m \to \infty$ it will be $\Delta t \to 0$ and:

$$\sum_{l=1}^{m}\varepsilon_i^{(l)}u(k_i^{(l)})\Delta t \xrightarrow[\Delta t \to 0]{} \int_0^t \varepsilon_i(x)u(k_i(x))dx,\ i=\overline{1,n}, \tag{11}$$

where: $k_i(x)$ - the number of messages in system S_i at the time moment x, $\varepsilon_i(x)$ - number of employed lines in the system messages S_i at the time moment x.

According to (10) and (11), we obtain the following approximate relationship:

$$v_i(t) = M\{V_i(t)\} = v_{i0} + v_i(t) = v_{i0} -$$
$$+\left[a_{0i}\lambda(t)p_{0i}\sum_{j=1}^{n}\psi_{ij}(t) - \mu_i(t)\left(b_{i0}\alpha_i(t) + \sum_{j=1}^{n}\beta_{ij}(t)\gamma_{ij}\right)\int_0^t \rho_i(x)dx + \overline{\eta}_i \right] t -$$
$$-\overline{\eta}\left[\sum_{i=2}^{n}\left(p_{0i}\sum_{j=1}^{n}\lambda(t)\psi_{ij}(t) + \left(\alpha_i(t) + \sum_{j=1}^{n}\beta_{ij}(t)\right)\mu_i(t)\int_0^t \rho_i(x)dx \right) - \right. \tag{12}$$
$$\left. -\sum_{i=1}^{n}\left[p_{0i}(\lambda(t) - \lambda(t)\varphi_i(t)) + \mu_i(t) - \mu_i(t)\beta_{ii}(t)\right]\right],\ i=\overline{1,n}.$$

To find the value of $\rho_i(t)$ you can use the technique, finding the average number of occupied lines in network systems, as described in [4].

Consider the special case when we have the HM-network without messages in [4] the open Multiline HM-network with heterogeneous messages. Mesages in the transition between systems network can change its type. Consider the case when the incomes from the state transition network are RV or functions that depend on them. Expressions were obtained for the expected incomes systems for each type of network messages. In the case of similar messages, the expressions for the expected incomes systems in the network, obtained in [4], took the form

$$\begin{gathered} v_i(t) = v_{i0} + \left[a_{0i}\left(1+\lambda(t)\left(p_{0i} - \sum_{i=1}^{n} p_{0i}\right)\int_0^t \rho_i(x)dx\right) + \overline{\eta}\right]t - \\ - \overline{\eta}\mu_i(t)\left(b_{i0}p_{i0} - \sum_{j=1}^{n} p_{ij}\gamma_{ij}\right)\int_0^t \rho_i(x)dx + \overline{\eta}\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i(t)p_{ij}\int_0^t \rho_i(x)dx,\ i=\overline{1,n}. \end{gathered} \tag{13}$$

In our case, $f^{(i)}(k,t)=1$ then the expression for the conditional probabilities $\varphi_i(t)$, $\psi_{ij}(t)$, $\alpha_i(t)$ and $\beta_{ij}(t)$. According to [1], there are as follows:

$$\begin{gathered} \varphi_i(t) = \left(1-f^{(i)}(t)\right)\left(p_{i0} + \sum_{j=1}^{n} p_{ij}\varphi_j(t)\right) = 0,\ i=\overline{1,n}, \\ \psi_{ij}(t) = f^{(i)}(k,t)\delta_{ij} + (1-f^{(i)}(t))\sum_{l=1}^{n} p_{il}\psi_{lj}(t) = \delta_{ij},\ i,j=\overline{1,n}, \\ \alpha_i(t) = p_{i0} + \sum_{j=1}^{n} p_{ij}\varphi_j(t) = p_{i0},\ i=\overline{1,n}, \\ \beta_{ij}(t) = \sum_{l=1}^{n} p_{il}\psi_{lj}(t) = \sum_{l=1}^{n}\delta_{lj}(t)p_{il},\ i,j=\overline{1,n}. \end{gathered} \tag{14}$$

Substituting them into (8), we obtain the expression (9). This confirms the correctness of the research and the results on an open HM-network with the same type messages bypass of systems in this case.

Example 2

Consider an open network $n=5$, $K=100$, other parameters as in Example 1. Let us leave all the initial data the same as in the Example 1. Let the service rates be equal $\mu_i(t) = \frac{\mu_i}{t}$, $i=\overline{1,5}$. Vector of the QS network looks like $m = (m_1, m_2, \ldots, m_n)$ $=(10, 30, 7, 1, 10)$. Random incomes systems are: $R_{ij}(\xi_i) = 3000\xi_i$, $i,j=\overline{1,5}$. Consider the time period of 24 hours, $t \in [0,T]$, $T = 24$. Mathematical expectations $M\{\eta_i\} = \overline{\eta}_i$, and $M\{R_{ij}(\xi_i)\} = \gamma_{ij}$, $i,j = \overline{1,n}$, are equals respectively: $\overline{\eta}_1 = 100$, $\overline{\eta}_2 = 200$, $\overline{\eta}_3 = \overline{\eta}_4 = \overline{\eta}_5 = 500$, $\gamma_{ij} = 3000\int_0^{24} t e^{-\mu_i t} dt = 864000 e^{-\mu_i}$.

Let at the initial moment $t_0 = 0$ incomes system be zero. Then, using the formula (7) and the package of mathematical calculations *Wolfram Mathematica*, expressions were obtained for the changes expected in the incomes systems S_i, $i = \overline{1,n}$, and the income of the whole network at the time interval. The graph of this function is shown at Figure 2, and the graph of the expected income of the whole network - in Figure 3.

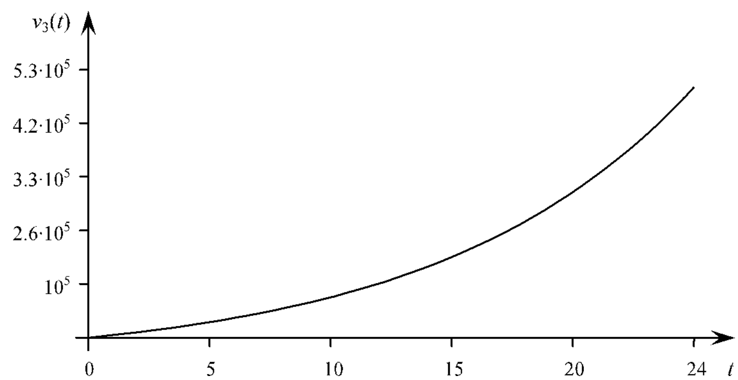

Fig. 2. The graph of change of expected income in the system S_3

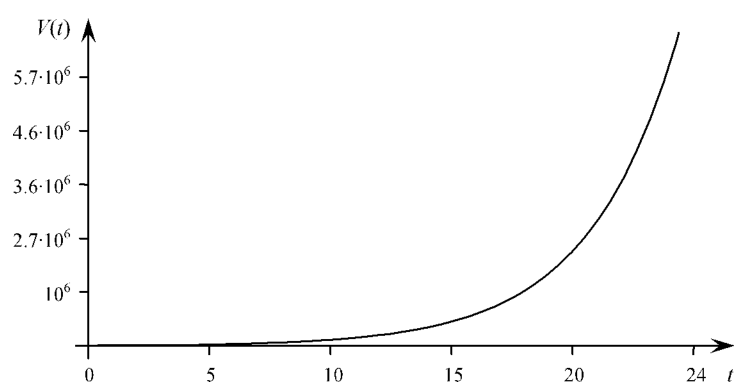

Fig. 3. The graph of change of expected income of the whole network

Conclusions

An open HM-network with one-type messages bypass of systems in the transient behaviour was investigated. Two cases were considered: when incomes from transitions between network states are deterministic functions depending on states and time, and network systems are single-line, and when incomes from transitions between network states are functions depending on the random variables.

The expected incomes of such a network were found in both cases on condition that the probabilities messages bypasses of systems network, parameters of incoming flow of messages and services depend on time. Two examples were considered and calculated.

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