Annales Mathematicae Silesianae **37** (2023), no. 1, 117–137
DOI: 10.2478/amsil-2023-0004

THE GENERALIZATION OF GAUSSIANS AND LEONARDO'S OCTONIONS

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Abstract. In order to explore the Leonardo sequence, the process of complexification of this sequence is carried out in this work. With this, the Gaussian and octonion numbers of the Leonardo sequence are presented. Also, the recurrence, generating function, Binet's formula, and matrix form of Leonardo's Gaussian and octonion numbers are defined. The development of the Gaussian numbers is performed from the insertion of the imaginary component i in the one-dimensional recurrence of the sequence. Regarding the octonions, the terms of the Leonardo sequence are presented in eight dimensions. Furthermore, the generalizations and inherent properties of Leonardo's Gaussians and octonions are presented.

1. Introduction

Historically, the origin of the Leonardo sequence reports that this sequence was possibly created by Leonardo of Pisa (1180-1250) [2]. This fact is due to its great similarity with the Fibonacci sequence and also because the sequence

Received: 11.01.2022. Accepted: 06.02.2023. Published online: 27.02.2023.
(2020) Mathematics Subject Classification: 11B37, 11B39.
Key words and phrases: matrix form, Binet's formula, Gaussian, octonions, Leonardo sequence.
The part of research development in Brazil had the financial support of the National Council for Scientific and Technological Development – CNPq and the Ceará Foundation for Support to Scientific and Technological Development (Funcap). The research development aspect in Portugal is financed by National Funds through FCT – Fundação para a Ciência e Tecnologia. I.P, within the scope of the UID/CED/00194/2020 project.

has the same name as the Italian mathematician. It is possible to find the mathematical evolution of this sequence in the works [3, 1, 7, 10, 8].

The Leonardo sequence satisfies the following recurrence relation:

$$L_n = L_{n-1} + L_{n-2} + 1, \quad n \geq 2, \quad L_0 = L_1 = 1.$$

On the other hand, for $n+1$ we can rewrite this recurrence relation as $L_{n+1} = L_n + L_{n-1} + 1$. With this, we can add the term $-L_{n+1}$, resulting in another recurrence relation. Hence, we have:

$$L_n - L_{n+1} = L_{n-1} + L_{n-2} + 1 - L_n - L_{n-1} - 1,$$

i.e.,

$$L_{n+1} = 2L_n - L_{n-2},$$

where $L_0 = L_1 = 1$ are initial conditions.

In 1981 Harman ([4]) introduced the Gaussian numbers denoted by $z = a + bi$ with a, $b \in \mathbb{Z}$ and $i^2 = -1$, which will be associated with the Leonardo sequence, improving the process of mathematical complexification of this sequence. Regarding octonion numbers, according to Vieira, Alves, and Catarino (2020) [9], it can be said that in the work of Karatas and Halici (2017) [5] the algebra in sixteen dimensions was studied and Horadam octonions by Horadam sequence which is a generalization of second order recurrence relations were defined.

Octonions were defined as the $\mathbb{R}$ numerical field, writing as ([6, 5]):

$$p = p' + p''\mathrm{e},$$

where $p', p'' \in H = \{a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \mid \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \boldsymbol{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$.

For the operation of adding and multiplying between two octonions, $p = p' + p''\mathrm{e}$ and, $q = q' + q''\mathrm{e}$:

$$p + q = (p' + q') + (p'' + q'')\mathrm{e},$$

$$pq = (p'q' - \overline{q''}p'') + (q''p' + p''\overline{q'})\mathrm{e},$$

where $\overline{q'}$ and $\overline{q''}$ are the conjugates of the quaternions q' and q'', respectively. Therefore, $\mathbb{O}$ is the algebra of the octonions, on a natural basis in the space over $\mathbb{R}$ formed by the elements: $\mathrm{e}_0 = 1$, $\mathrm{e}_1 = \boldsymbol{i}$, $\mathrm{e}_2 = \boldsymbol{j}$, $\mathrm{e}_3 = \boldsymbol{k}$, $\mathrm{e}_4 = \mathrm{e}$, $\mathrm{e}_5 = \boldsymbol{i}\mathrm{e}$, $\mathrm{e}_6 = \boldsymbol{j}\mathrm{e}$, $\mathrm{e}_7 = \boldsymbol{k}\mathrm{e}$ and the multiplication of these numbers is shown in the Table 1.

Table 1. Multiplication of the octonions of $\mathbb{O}$.
Source: [6]

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Thus, the following notation is used for octonions:

$$p = \sum_{p=0}^{7} p_s \mathrm{e}_\mathrm{s},$$

where p_s is the real coefficient, with $p \in \mathbb{O}$, in format $p = \mathrm{Re}(p) + \mathrm{Im}(p)$, where $\mathrm{Re}(p) = p_0$ represents the real part and, $\mathrm{Im}(p) = \sum_{s=1}^{7} p_s \mathrm{e}_\mathrm{s}$ represents the imaginary part.

Therefore, in this work, it is intended to continue the mathematical evolution of this sequence, presenting the Gaussian and octonion numbers of Leonardo's sequence.

2. Leonardo's Gaussians

Henceforward, Leonardo's Gaussian numbers will be introduced, beginning complex studies around this sequence, with the insertion of an imaginary unit. Thus, their respective mathematical aspects are portrayed.

DEFINITION 2.1. For $n \geqslant 0$, *Leonardo's Gaussians* are defined by:

$$GL_n = L_n + iL_{n+1},$$

where $L_0 = L_1 = 1$. In particular, $GL_0 = 1 + i, GL_1 = 1 + 3i$.

From the previous definition, it is easy to see that for all $n \geqslant 3$ and $n \in \mathbb{N}$, the recurrence formula of Leonardo's Gaussian is given by:

$$GL_n = 2GL_{n-1} - GL_{n-3},$$

where $GL_0 = 1 + i$ and $GL_1 = 1 + 3i$.

Definition 2.2. For $n \geqslant 0$, *Fibonacci's Gaussians* are defined by:

$$GF_n = F_n + iF_{n+1},$$

where $F_0 = 0$, $F_1 = 1$. In particular, $GF_0 = i$ and $GF_1 = 1 + i$.

From the previous definition, it is easy to see that for all $n \geqslant 2$ and $n \in \mathbb{N}$, the recurrence formula of Fibonacci's Gaussian is given by:

$$GF_n = GF_{n-1} + GF_{n-2},$$

where $GF_0 = i$ and $GF_1 = 1 + i$.

Definition 2.3. Leonardo and Fibonacci's *Gaussian recurrence formula* is given by:

$$GLF_n = GL_n + GF_n,$$

where $n \in \mathbb{Z}$.

Theorem 2.4. *The generating function of Leonardo's Gaussians is given by:*

$$g(GL_n, x) = \frac{1 + i + (1 - i)(-x + x^2)}{(1 - 2x - x^3)}.$$

Proof. Let us consider the function

$$g(GL_n, x) = GL_0 + GL_1 x + GL_2 x^2 + \ldots + GL_n x^n + \ldots.$$

Multiplying this function by $2x$ and x^3, we get

$$2xg(GL_n, x) = 2GL_0 x + 2GL_1 x^2 + 2GL_2 x^3 + \ldots + 2GL_{n-1} x^n + \ldots,$$

$$x^3 g(GL_n, x) = GL_0 x^3 + GL_1 x^4 + GL_2 x^5 + \ldots + GL_{n-3} x^n + \ldots.$$

Subtracting the previous equalities and after some calculations, we obtain:

$$(1 - 2x - x^3)g(GL_n, x) = GL_0 + (GL_1 - 2GL_0)x + (GL_2 - 2GL_1)x^2,$$

$$(1 - 2x - x^3)g(GL_n, x) = 1 + i - (1 - i)x + (1 - i)x^2,$$

$$(1 - 2x - x^3)g(GL_n, x) = 1 + i + (1 - i)(-x + x^2),$$

$$g(GL_n, x) = \frac{1 + i + (1 - i)(-x + x^2)}{(1 - 2x - x^3)}.$$

□

THEOREM 2.5. *The Binet formula of Leonardo's Gaussians, with $n \in \mathbb{Z}$, is:*

$$GL_n = A_g(1+ir_1)r_1^n + B_g(1+ir_2)r_2^n + C_g(1+ir_3)r_3^n,$$

where r_1, r_2, r_3 are the roots of the characteristic polynomial $r^3 - 2r^2 + 1 = 0$,

$$A_g = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, \quad B_g = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)},$$

$$C_g = \frac{(r_1-1)(r_2-1)}{(r_3-r_1)(r_3-r_2)}.$$

PROOF. Through the Binet's formula $GL_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$ and the recurrence of Leonardo's Gaussians $GL_n = L_n + iL_{n+1}$, with the initial values $GL_0 = 1+i$, $GL_1 = 1+3i$ and $GL_2 = 3+5i$, it is possible to obtain the following system of equations:

$$\begin{cases} \alpha + \beta + \gamma = 1 + i, \\ \alpha r_1 + \beta r_2 + \gamma r_3 = 1 + 3i, \\ \alpha r_1^2 + \beta r_2^2 + \gamma r_3^2 = 3 + 5i. \end{cases}$$

Solving the system, we get:

$$\alpha = \frac{(3+5i) + (-r_2 - r_3)(1+3i) + r_2 r_3(1+i)}{r_1^2 - r_1 r_2 - r_1 r_3 + r_2 r_3},$$

$$\beta = \frac{(3+5i) + (-r_1 - r_3)(1+3i) + r_1 r_3(1+i)}{r_2^2 - r_2 r_3 - r_1 r_2 + r_1 r_3},$$

$$\gamma = \frac{(3+5i) + (-r_1 - r_2)(1+3i) + r_1 r_2(1+i)}{r_3^2 + r_1 r_2 - r_1 r_3 - r_2 r_3}.$$

Through Girard's relations: $r_1 r_2 r_3 = -1, r_1 + r_2 + r_3 = 2$ and $r_1 r_2 + r_2 r_3 + r_1 r_3 = 0$, it is easy to see that:

$$\alpha = \frac{(r_2 r_2 - r_2 - r_3 + 1)}{(r_1 - r_2)(r_1 - r_3)}(1+ir_1) = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}(1+ir_1) = A_g(1+ir_1),$$

$$\beta = \frac{(r_1 r_3 - r_1 - r_3 + 1)}{(r_2 - r_1)(r_2 - r_3)}(1+ir_2) = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}(1+ir_2) = B_g(1+ir_2),$$

$$\gamma = \frac{(r_1 r_2 - r_1 - r_2 + 1)}{(r_3 - r_1)(r_3 - r_2)}(1+ir_3) = \frac{(r_1-1)(r_2-1)}{(r_3-r_1)(r_3-r_2)}(1+ir_3) = C_g(1+ir_3). \square$$

Based on the work of Vieira, Mangueira, Alves and Catarino ([10]), one can establish the matrix form of Leonardo's sequence in the complex form.

Proposition 2.6. *For $n \geqslant 2$ and $n \in \mathbb{N}$, the matrix form of Leonardo's Gaussians is given by:*

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} GF_2 & GF_0 & \frac{GLF_{-2}}{L_{n+2}} \\ \frac{GLF_{-2}}{L_{n+1}} & GF_{-1} & GF_0 \\ -GF_0 & \frac{GLF_{-2}}{L_n} & GF_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} L_{n+2} & L_{n+1} & L_n \end{bmatrix} \begin{bmatrix} GF_2 & GF_0 & \frac{GLF_{-2}}{L_{n+2}} \\ \frac{GLF_{-2}}{L_{n+1}} & GF_{-1} & GF_0 \\ -GF_0 & \frac{GLF_{-2}}{L_n} & GF_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} GL_{n+2} & GL_{n+1} & GL_n \end{bmatrix},$$

where $GLF_n = GL_n + GF_n$, for $n < 0$.

Proof. By the principle of finite induction, we have for $n = 2$:

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} GF_2 & GF_0 & \frac{GLF_{-2}}{L_4} \\ \frac{GLF_{-2}}{L_3} & GF_{-1} & GF_0 \\ -GF_0 & \frac{GLF_{-2}}{L_2} & GF_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 & 3 \end{bmatrix} \begin{bmatrix} GF_2 & GF_0 & \frac{GLF_{-2}}{9} \\ \frac{GLF_{-2}}{5} & GF_{-1} & GF_0 \\ -GF_0 & \frac{GLF_{-2}}{3} & GF_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 9GF_2 - 3GF_0 + GLF_{-2} & 9GF_0 + 5GF_{-1} + GLF_{-2} & GLF_{-2} + 5GF_0 + 3GF_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} GL_4 & GL_3 & GL_2 \end{bmatrix}.$$

So, assume it is true for any $n = k$, with $k \in \mathbb{N}$:

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} GF_2 & GF_0 & \frac{GLF_{-2}}{L_{k+2}} \\ \frac{GLF_{-2}}{L_{k+1}} & GF_{-1} & GF_0 \\ -GF_0 & \frac{GLF_{-2}}{L_k} & GF_{-1} \end{bmatrix} = \begin{bmatrix} GL_{k+2} & GL_{k+1} & GL_k \end{bmatrix}.$$

Let us show that it is still valid for $n = k+1$:

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}^{k+1}\begin{bmatrix}GF_2 & GF_0 & \frac{GLF_{-2}}{L_{k+3}}\\ \frac{GLF_{-2}}{L_{k+2}} & OF_{-1} & GF_0\\ -GF_0 & \frac{GLF_{-2}}{L_{k+1}} & GF_{-1}\end{bmatrix}$$

$$= \begin{bmatrix}L_{k+2} & L_{k+1} & L_k\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}GF_2 & GF_0 & \frac{GLF_{-2}}{L_{k+3}}\\ \frac{GLF_{-2}}{L_{k+2}} & GF_{-1} & GF_0\\ -GF_0 & \frac{GLF_{-2}}{L_{k+1}} & GF_{-1}\end{bmatrix}$$

$$= \begin{bmatrix}L_{k+3} & L_{k+2} & L_{k+1}\end{bmatrix}\begin{bmatrix}GF_2 & GF_0 & \frac{GLF_{-2}}{L_{k+3}}\\ \frac{GLF_{-2}}{L_{k+2}} & GF_{-1} & GF_0\\ -GF_0 & \frac{GLF_{-2}}{L_{k+1}} & GF_{-1}\end{bmatrix}$$

$$= [L_{k+3}GF_2 - L_{k+1}GF_0 + GLF_{-2} \quad L_{k+3}GF_0 + L_{k+2}GF_{-1} + GLF_{-2} \quad L_{k+2}GF_0 + L_{k+1}GF_{-1} + GLF_{-2}]$$

$$= \begin{bmatrix}GL_{k+3} & GL_{k+2} & GL_{k+1}\end{bmatrix}. \qquad \square$$

3. The generalization of Leonardo's Gaussians

Next, the behavior of terms with non-positive integer indices of Leonardo's Gaussians will be analyzed.

Definition 3.1. For all $n > 0$ and $n \in \mathbb{N}$, *Leonardo's Gaussians, for non-positive integer index*, are defined by the equation:

$$GL_{-n} = \sum_{s=0}^{7} L_{-n+s}\mathrm{e}_s.$$

From the previous definition, it is easy to see that for all $n > 0$ and $n \in \mathbb{N}$, the recurrence formula of Leonardo's Gaussians for non-positive integer index, is given by:

$$GL_{-n} = 2GL_{-n+2} - GL_{-n+3},$$

where $GL_0 = 1+i, GL_1 = 1+3i$ and $GL_2 = 4+6i$.

Proposition 3.2. *The generating function of Leonardo's Gaussians for non-positive integer index, is expressed by:*

$$g(GL_{-n},x)=\frac{1+i+(-1+i)x+(-1-3i)x^2}{x^3-2x^2+1}.$$

Proof. Let us consider the function

$$g(GL_{-n,x})=\sum_{n=0}^{\infty}GL_{-n}x^n=GL_0+GL_{-1}x+GL_{-2}x^2+\ldots+GL_{-n}x^n+\ldots.$$

Multiplying this function by $2x^2$ and x^3, we have:

$$2x^2g(GL_{-n},x)=2GL_0x^2+2GL_{-1}x^3+2GL_{-2}x^4+\ldots+2GL_{-n-2}x^n+\ldots,$$

$$x^3g(GL_{-n},x)=GL_0x^3+GL_{-1}x^4+GL_{-2}x^5+\ldots+GL_{-n-3}x^n+\ldots.$$

Subtracting the previous equalities and after some calculations, we obtain:

$$(x^3-2x^2+1)g(GL_{-n},x)=GL_0+GL_{-1}x+(-2GL_0+GL_{-2})x^2,$$

$$(x^3-2x^2+1)g(GL_{-n},x)=1+i+(-1+i)x+(-1-3i)x^2,$$

$$g(GL_{-n},x)=\frac{1+i+(-1+i)x+(-1-3i)x^2}{x^3-2x^2+1}.$$ □

Proposition 3.3. *For $n>0$ and $n\in\mathbb{N}$, the generating matrix of Leonardo's Gaussians, with a non-positive integer index, is given by:*

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 1 & 0 & 2\\ 0 & 1 & 0\end{bmatrix}^n\begin{bmatrix}GF_2 & GF_0 & \frac{GLF_{-2}}{L_{-n+2}}\\ \frac{GLF_{-2}}{L_{-n+1}} & GF_{-1} & GF_0\\ -GF_0 & \frac{GLF_{-2}}{L_{-n}} & GF_{-1}\end{bmatrix}$$

$$=\begin{bmatrix}L_{-n+2} & L_{-n+1} & L_{-n}\end{bmatrix}\begin{bmatrix}GF_2 & GF_0 & \frac{GLF_{-2}}{L_{-n+2}}\\ \frac{GLF_{-2}}{L_{-n+1}} & GF_{-1} & GF_0\\ -GF_0 & \frac{GLF_{-2}}{L_{-n}} & GF_{-1}\end{bmatrix}$$

$$=\begin{bmatrix}GL_{-n+2} & GL_{-n+1} & GL_{-n}\end{bmatrix},$$

where $GLF_{-n}=GL_{-n}+GF_{-n}$, $GF_{-2}=i-1$, $GF_{-1}=1$ and $GF_0=i$.

Proof. Similarly to the demonstration performed in Proposition 2.6, this property can be validated. □

4. Leonardo's octonions

In this section, Leonardo's octonions will be studied, addressing their respective mathematical properties.

Definition 4.1. For $n \geqslant 0$, *Leonardo's octonions* are defined by:

$$OL_n = \sum_{s=0}^{7} L_{n+s}\mathrm{e_s},$$

where $OL_0 = \sum_{s=0}^{7} L_s\mathrm{e_s}$, $OL_1 = \sum_{s=0}^{7} L_{1+s}\mathrm{e_s}$, $OL_2 = \sum_{s=0}^{7} L_{2+s}\mathrm{e_s}$.

From the previous definition, it is easy to see that for all $n \geqslant 3$ and $n \in \mathbb{N}$, the recurrence formula of Leonardo's octonions is given by:

$$OL_n = 2OL_{n-1} - OL_{n-3},$$

where $OL_0 = \sum_{s=0}^{7} L_s\mathrm{e_s}$, $OL_1 = \sum_{s=0}^{7} L_{1+s}\mathrm{e_s}$, $OL_2 = \sum_{s=0}^{7} L_{2+s}\mathrm{e_s}$.

Theorem 4.2. *The generating function of Leonardo's octonions, OL_n, is given by:*

$$g(OL_n, x) = \frac{1}{(1-2x-x^3)} \sum_{s=0}^{7} (L_s + L_{s-2}x + L_{s-1}x^2)\mathrm{e_s}.$$

Proof. Let us consider the function

$$g(OL_n, x) = OL_0 + OL_1 x + OL_2 x^2 + \ldots + OL_n x^n + \ldots.$$

Multiplying the function by $2x$ and x^3, we get:

$$2xg(OL_n, x) = 2OL_0 x + 2OL_1 x^2 + 2OL_2 x^3 + \ldots + 2OL_{n-1} x^n + \ldots,$$

$$x^3 g(OL_n, x) = OL_0 x^3 + OL_1 x^4 + OL_2 x^5 + \ldots + OL_{n-3} x^n + \ldots.$$

Subtracting the previous equalities and after some calculations, we obtain:

$$(1-2x-x^3)g(OL_n, x) = OL_0 + (OL_1 - 2OL_0)x + (OL_2 - 2OL_1)x^2,$$

$$g(OL_n, x) = \frac{1}{(1-2x-x^3)} \sum_{s=0}^{7} (L_s + L_{s-2}x + L_{s-1}x^2)\mathrm{e_s}. \qquad \square$$

Theorem 4.3. *Binet's formula for Leonardo's octonions, with $n \in \mathbb{Z}$, is given by:*

$$OL_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n,$$

where r_1, r_2 and r_3 are the roots of the characteristic polynomial $r^3 - 2r^2 + 1 = 0$,

$$A_l = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, \quad B_l = \frac{(r_1-1)(r_3-1)}{(r_2-x_1)(r_2-r_3)}, \quad C_l = \frac{(r_1-1)(r_2-1)}{(r_3-r_1)(r_3-r_2)},$$

$$\alpha_{ol} = \sum_{s=0}^{7} r_1^s \mathrm{e_s}, \quad \beta_{ol} = \sum_{s=0}^{7} r_2^s \mathrm{e_s}, \quad \gamma_{ol} = \sum_{s=0}^{7} r_3^s \mathrm{e_s},$$

$$\alpha = A_l \alpha_{ol}, \quad \beta = B_l \beta_{ol}, \quad \gamma = C_l \gamma_{ol}.$$

Proof. Through the Binet's formula $oL_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$ and the recurrence of Leonardo's octonions $OL_n = \sum_{s=0}^{7} L_{n+s}\mathrm{e_s}$, with the initial values $OL_0 = \sum_{s=0}^{7} L_s \mathrm{e_s}$, $OL_1 = \sum_{s=0}^{7} L_{s+1}\mathrm{e_s}$ and $OL_2 = \sum_{s=0}^{7} L_{s+2}\mathrm{e_s}$, it is possible to obtain the following system of equations:

$$\begin{cases} \alpha + \beta + \gamma = \sum_{s=0}^{7} L_s \mathrm{e_s}, \\ \alpha r_1 + \beta r_2 + \gamma r_3 = \sum_{s=0}^{7} L_{s+1} \mathrm{e_s}, \\ \alpha r_1^2 + \beta r_2^2 + \gamma r_3^2 = \sum_{s=0}^{7} L_{s+2} \mathrm{e_s}. \end{cases}$$

Solving this system, we have:

$$\alpha = \frac{\Big(\sum_{s=0}^{7} L_{s+2}\mathrm{e_s}\Big) + (-r_2 - r_3)\Big(\sum_{s=0}^{7} L_{s+1}\mathrm{e_s}\Big) + r_2 r_3 \Big(\sum_{s=0}^{7} L_s \mathrm{e_s}\Big)}{r_1^2 - r_1 r_2 - r_1 r_3 + r_2 r_3},$$

$$\beta = \frac{\Big(\sum_{s=0}^{7} L_{s+2}\mathrm{e_s}\Big) + (-r_1 - r_3)\Big(\sum_{s=0}^{7} L_{s+1}\mathrm{e_s}\Big) + r_1 r_3 \Big(\sum_{s=0}^{7} L_s \mathrm{e_s}\Big)}{r_2^2 - r_2 r_3 - r_1 r_2 + r_1 r_3},$$

$$\gamma = \frac{\Big(\sum_{s=0}^{7} L_{s+2}\mathrm{e_s}\Big) + (-r_1 - r_2)\Big(\sum_{s=0}^{7} L_{s+1}\mathrm{e_s}\Big) + r_1 r_2 \Big(\sum_{s=0}^{7} L_s \mathrm{e_s}\Big)}{r_3^2 + r_1 r_2 - r_1 r_3 - r_2 r_3}.$$

Through Girard's relations: $r_1x_2r_3 = -1, r_1 + r_2 + r_3 = 2$ and $r_1r_2 + r_2r_3 + r_1r_3 = 0$, it is easy to see that:

$$\alpha = \frac{(r_2r_2 - r_2 - r_3 + 1)}{(r_1 - r_2)(r_1 - r_3)} \sum_{s=0}^{7} r_1^s \mathrm{e_s} = \frac{(r_2 - 1)(r_3 - 1)}{(r_1 - r_2)(r_1 - r_3)} \sum_{s=0}^{7} r_1^s \mathrm{e_s} = A_l \sum_{s=0}^{7} r_1^s \mathrm{e_s},$$

$$\beta = \frac{(r_1r_3 - r_1 - r_3 + 1)}{(r_2 - r_1)(r_2 - r_3)} \sum_{s=0}^{7} r_2^s \mathrm{e_s} = \frac{(r_1 - 1)(r_3 - 1)}{(r_2 - r_1)(r_2 - r_3)} \sum_{s=0}^{7} r_2^s \mathrm{e_s} = B_l \sum_{s=0}^{7} r_2^s \mathrm{e_s},$$

$$\gamma = \frac{(r_1r_2 - r_1 - r_2 + 1)}{(r_3 - r_1)(r_3 - r_2)} \sum_{s=0}^{7} r_3^s \mathrm{e_s} = \frac{(r_1 - 1)(r_2 - 1)}{(r_3 - r_1)(r_3 - r_2)} \sum_{s=0}^{7} r_3^s \mathrm{e_s} = C_l \sum_{s=0}^{7} r_3^s \mathrm{e_s}.$$

Defining $\alpha_{ol} = \sum_{s=0}^{7} r_1^s \mathrm{e_s}$, $\beta_{ol} = \sum_{s=0}^{7} r_2^s \mathrm{e_s}$ and $\gamma_{ol} = \sum_{s=0}^{7} r_3^s \mathrm{e_s}$, it is easy to see that:

$$\alpha = A_l\alpha_{ol}, \quad \beta = B_l\beta_{ol}, \quad \gamma = C_l\gamma_{ol}. \qquad \square$$

The matrix form of Leonardo's octonions is based on the work of Vieira, Mangueira, Alves and Catarino ([10]), in which we found a development on the matrix form of the one-dimensional Leonardo sequence.

PROPOSITION 4.4. *For $n \geqslant 2$ and $n \in \mathbb{N}$, the matrix form of Leonardo's octonions is given by:*

$$\begin{aligned}
\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} OF_2 & OF_0 & \frac{OLF_{-2}}{L_{n+2}} \\ \frac{OLF_{-2}}{L_{n+1}} & OF_{-1} & OF_0 \\ -OF_0 & \frac{OLF_{-2}}{L_n} & OF_{-1} \end{bmatrix}
&= \begin{bmatrix} L_{n+2} & L_{n+1} & L_n \end{bmatrix} \begin{bmatrix} OF_2 & OF_0 & \frac{OLF_{-2}}{L_{n+2}} \\ \frac{OLF_{-2}}{L_{n+1}} & OF_{-1} & OF_0 \\ -OF_0 & \frac{OLF_{-2}}{L_n} & OF_{-1} \end{bmatrix} \\
&= \begin{bmatrix} OL_{n+2} & OL_{n+1} & OL_n \end{bmatrix},
\end{aligned}$$

where $OLF_n = OL_n + OF_n$.

PROOF. By the principle of finite induction, we have for $n = 2$:

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}^2\begin{bmatrix}OF_2 & OF_0 & \frac{OLF_{-2}}{L_4}\\ \frac{OLF_{-2}}{L_3} & OF_{-1} & OF_0\\ -OF_0 & \frac{OLF_{-2}}{L_2} & OF_{-1}\end{bmatrix}$$

$$= \begin{bmatrix}9 & 5 & 3\end{bmatrix}\begin{bmatrix}OF_2 & OF_0 & \frac{OLF_{-2}}{9}\\ \frac{OLF_{-2}}{5} & OF_{-1} & OF_0\\ -OF_0 & \frac{OLF_{-2}}{3} & OF_{-1}\end{bmatrix}$$

$$= \begin{bmatrix}9OF_2-3OF_0+OLF_{-2} & 9OF_0+5OF_{-1}+OLF_{-2} & OLF_{-2}+5OF_0+3OF_{-1}\end{bmatrix}$$

$$= \begin{bmatrix}OL_4 & OL_3 & OL_2\end{bmatrix}.$$

So, assume it is true for any $n = k$, with $k \in \mathbb{N}$:

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}^k\begin{bmatrix}OF_2 & OF_0 & \frac{OLF_{-2}}{L_{k+2}}\\ \frac{OLF_{-2}}{L_{k+1}} & OF_{-1} & OF_0\\ -OF_0 & \frac{OLF_{-2}}{L_k} & OF_{-1}\end{bmatrix} = \begin{bmatrix}OL_{k+2} & OL_{k+1} & OL_k\end{bmatrix}.$$

Finally, the validity for $n = k + 1$ is verified:

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}^{k+1}\begin{bmatrix}OF_2 & OF_0 & \frac{OLF_{-2}}{L_{k+3}}\\ \frac{OLF_{-2}}{L_{k+2}} & OF_{-1} & OF_0\\ -OF_0 & \frac{OLF_{-2}}{L_{k+1}} & OF_{-1}\end{bmatrix}$$

$$= \begin{bmatrix}L_{k+2} & L_{k+1} & L_k\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}OF_2 & OF_0 & \frac{OLF_{-2}}{L_{k+3}}\\ \frac{OLF_{-2}}{L_{k+2}} & OF_{-1} & OF_0\\ -OF_0 & \frac{OLF_{-2}}{L_{k+1}} & OF_{-1}\end{bmatrix}$$

$$= \begin{bmatrix}L_{k+3} & L_{k+2} & L_{k+1}\end{bmatrix}\begin{bmatrix}OF_2 & OF_0 & \frac{OLF_{-2}}{L_{k+3}}\\ \frac{OLF_{-2}}{L_{k+2}} & OF_{-1} & OF_0\\ -OF_0 & \frac{OLF_{-2}}{L_{k+1}} & OF_{-1}\end{bmatrix}$$

$$= [L_{k+3}OF_{-2}-L_{k+1}OF_0+OLF_{-2} \quad L_{k+3}OF_0+L_{k+2}OF_{-1}+OLF_{-2}$$

$$L_{k+2}OF_0+L_{k+1}OF_{-1}+OLF_{-2}]$$

$$= \begin{bmatrix}OL_{k+3} & OL_{k+2} & OL_{k+1}\end{bmatrix}.$$

□

5. The generalization of Leonardo's octonions

Next, the behavior of terms with non-positive integer indices of Leonardo's octonions will be analyzed.

DEFINITION 5.1. For all $n > 0$ and $n \in \mathbb{N}$, *Leonardo's octonions, for non-positive integer index*, are defined by the equation:

$$OL_{-n} = \sum_{s=0}^{7} L_{-n+s}\mathrm{e_s}.$$

From the previous definition, it is easy to see that for all $n > 0$ and $n \in \mathbb{N}$, the recurrence formula of Leonardo's octonions for non-positive integer index, is given by:

$$OL_{-n} = 2OL_{-n+2} - OL_{-n+3},$$

where $OL_{-1} = \sum_{s=0}^{7} L_{-1+s}\mathrm{e_s}, OL_{-2} = \sum_{s=0}^{7} L_{-2+s}\mathrm{e_s}, OL_{-3} = \sum_{s=0}^{7} L_{-3+s}\mathrm{e_s}$.

PROPOSITION 5.2. *The generating function of Leonardo's octonions for non-positive integer index, is expressed by:*

$$g(OL_{-n}, x) = \frac{OL_0 + OL_{-1}x + (-2OL_0 + OL_{-2})x^2}{x^3 - 2x^2 + 1},$$

with the respective initial values:

$$OL_{-2} = \sum_{s=0}^{7} L_{-2+s}\mathrm{e_s}, \quad OL_{-1} = \sum_{s=0}^{7} L_{-2+s}\mathrm{e_s} \quad \textit{and} \quad OL_0 = \sum_{s=0}^{7} L_s\mathrm{e_s}.$$

PROOF. Let us consider the function

$$g(OL_{-n,x}) = \sum_{n=0}^{\infty} OL_{-n}x^n = OL_0 + OL_{-1}x + OL_{-2}x^2 + \ldots + OL_{-n}x^n + \ldots.$$

Multiplying the function by $2x^2$ and x^3, we have:

$$2x^2 g(OL_{-n}, x) = 2OL_0x^2 + 2OL_{-1}x^3 + 2OL_{-2}x^4 + \ldots + 2OL_{-n-2}x^n + \ldots,$$

$$x^3 g(OL_{-n}, x) = OL_0x^3 + OL_{-1}x^4 + OL_{-2}x^5 + \ldots + OL_{-n-3}x^n + \ldots.$$

Subtracting the previous equalities and after some calculations, we obtain:

$$(x^3 - 2x^2 + 1)g(OL_{-n}, x) = OL_0 + OL_{-1}x + (-2OL_0 + OL_{-2})x^2,$$

$$g(OL_{-n}, x) = \frac{OL_0 + OL_{-1}x + (-2OL_0 + OL_{-2})x^2}{x^3 - 2x^2 + 1}.$$

Note that $OL_{-2} = \sum_{s=0}^{7} L_{-2+s}\mathrm{e_s}$, $OL_{-1} = \sum_{s=0}^{7} L_{-2+s}\mathrm{e_s}$ and $OL_0 = \sum_{s=0}^{7} L_s\mathrm{e_s}$. □

PROPOSITION 5.3. *For $n > 0$ and $n \in \mathbb{N}$, the generator matrix of Leonardo's octonions, with non-positive integer index, is given by:*

$$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} OF_2 & OF_0 & \frac{OLF_{-2}}{L_{-n+2}} \\ \frac{OLF_{-2}}{L_{-n+1}} & OF_{-1} & OF_0 \\ -OF_0 & \frac{OLF_{-2}}{L_{-n}} & OF_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} L_{-n+2} & L_{-n+1} & L_{-n} \end{bmatrix} \begin{bmatrix} OF_2 & OF_0 & \frac{OLF_{-2}}{L_{-n+2}} \\ \frac{OLF_{-2}}{L_{-n+1}} & OF_{-1} & OF_0 \\ -OF_0 & \frac{OLF_{-2}}{L_{-n}} & OF_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} OL_{-n+2} & OL_{-n+1} & OL_{-n} \end{bmatrix},$$

where $OLF_{-n} = OL_{-n} + OF_{-n}$.

PROOF. Similar to the demonstration performed in Proposition 4.4, this property can be validated. □

6. Leonardo's Gaussians and octonions properties

Next, some properties inherent to Leonardo's Gaussians and octonions are studied.

PROPOSITION 6.1. *The sum of the first n numbers of Leonardo's Gaussians is given by:*

$$\sum_{m=3}^{n} GL_m = 2GL_{n-2} + 2GL_{n-1} - (2 + 4i) + \sum_{s=2}^{n-3} GL_s.$$

PROOF. Using the recurrence relation of Leonardo's Gaussians with $n \in \mathbb{N}$, we have:

$$GL_n = 2GL_{n-1} - GL_{n-3}. \tag{6.1}$$

Thus, evaluating the relation given in (6.1) in values of $n \geq 3$, we get:

$$\begin{aligned}
GL_3 &= 2GL_2 - GL_0,\\
GL_4 &= 2GL_3 - GL_1,\\
GL_5 &= 2GL_4 - GL_2,\\
GL_6 &= 2GL_5 - GL_3,\\
GL_7 &= 2GL_6 - GL_4,\\
&\vdots\\
GL_{n-2} &= 2GL_{n-3} - GL_{n-5},\\
GL_{n-1} &= 2GL_{n-2} - GL_{n-4},\\
GL_n &= 2GL_{n-1} - GL_{n-3}.
\end{aligned}$$

Through successive cancellations, we obtain:

$$\begin{aligned}
\sum_{m=3}^{n} GL_m &= GL_2 - GL_0 + GL_3 - GL_1 + GL_4\\
&\quad + \cdots + GL_{n-3} + 2GL_{n-2} + 2GL_{n-1}\\
&= 2GL_{n-2} + 2GL_{n-1} - (GL_0 + GL_1) + \sum_{s=2}^{n-3} GL_s.
\end{aligned}$$

□

PROPOSITION 6.2. *The sum of the numbers with even indexes of Leonardo's Gaussians is given by:*

$$\sum_{m=3}^{n} GL_{2m} = 2GL_{2n-1} - GL_1 + \sum_{s=1}^{n-2} GL_{2s+1}.$$

PROOF. Using the recurrence relation of Leonardo's Gaussians with $n \in \mathbb{N}$, we have:

$$GL_n = 2GL_{n-1} - GL_{n-3}.$$

Thus, evaluating the recurrence relation in values of $n \geq 3$, we get:

$$\begin{aligned} GL_4 &= 2GL_3 - GL_1, \\ GL_6 &= 2GL_5 - GL_3, \\ GL_8 &= 2GL_7 - GL_5, \\ &\vdots \\ GL_{2n-2} &= 2GL_{2n-3} - GL_{2n-5}, \\ GL_{2n} &= 2GL_{2n-1} - GL_{2n-3}. \end{aligned}$$

Through successive cancellations, we obtain:

$$\begin{aligned} \sum_{m=2}^{n} GL_{2m} &= GL_3 - GL_1 + GL_5 + \cdots + GL_{2n-3} + 2GL_{2n-1} \\ &= 2GL_{2n-1} - GL_1 + \sum_{s=1}^{n-2} GL_{2s+1}. \end{aligned}$$ □

PROPOSITION 6.3. *The sum of the odd index numbers of Leonardo's Gaussians is given by:*

$$\sum_{m=2}^{n} GL_{2m-1} = 2GL_{2n-2} - GL_0 + \sum_{s=0}^{n-3} GL_{2s}.$$

PROOF. Using the recurrence relation of Leonardo's Gaussians with $n \in \mathbb{N}$, we have:

$$GL_n = 2GL_{n-1} - GL_{n-3}.$$

Thus, evaluating the recurrence relation in values of $n \geq 3$, we get:

$$\begin{aligned} GL_3 &= 2GL_2 - GL_0, \\ GL_5 &= 2GL_4 - GL_2, \\ GL_7 &= 2GL_6 - GL_4, \\ &\vdots \\ GL_{2n-3} &= 2GL_{2n-4} - GL_{2n-6}, \\ GL_{2n-1} &= 2GL_{2n-2} - GL_{2n-4}. \end{aligned}$$

Through successive cancellations, we obtain:

$$\sum_{m=2}^{n} GL_{2m-1} = GL_2 - GL_0 + GL_4 + \cdots + GL_{2n-4} + 2GL_{2n-2}$$
$$= 2GL_{2n-2} - GL_0 + \sum_{s=0}^{n-3} GL_{2s}. \qquad \square$$

PROPOSITION 6.4. *The sum of the first n numbers of Leonardo's octonions is given by:*

$$\sum_{m=3}^{n} OL_m = 2OL_{n-2} + 2OL_{n-1} - \sum_{s=0}^{7}(L_s + L_{s+1})\mathrm{e_s} + \sum_{s=2}^{n-3} OL_s.$$

PROOF. Using the recurrence relation of Leonardo's octonions with $n \in \mathbb{N}$, we have:

$$OL_n = 2OL_{n-1} - OL_{n-3}. \tag{6.2}$$

Thus, evaluating the relation given in (6.2) in values of $n \geq 3$, we get:

$$OL_3 = 2OL_2 - OL_0,$$
$$OL_4 = 2OL_3 - OL_1,$$
$$OL_5 = 2OL_4 - OL_2,$$
$$OL_6 = 2OL_5 - OL_3,$$
$$OL_7 = 2OL_6 - OL_4,$$
$$\vdots$$
$$OL_{n-2} = 2OL_{n-3} - OL_{n-5},$$
$$OL_{n-1} = 2OL_{n-2} - OL_{n-4},$$
$$OL_n = 2OL_{n-1} - OL_{n-3}.$$

Through successive cancellations, we obtain:

$$\sum_{m=3}^{n} OL_m = OL_2 - OL_0 + OL_3 - OL_1 + OL_4 + \cdots + 2OL_{n-2} + 2OL_{n-1}$$
$$= 2OL_{n-2} + 2OL_{n-1} - (OL_0 + OL_1) + \sum_{s=2}^{n-3} OL_s.$$

Considering the initial values through Definition 4.1, it is concluded that:

$$2OL_{n-2} + 2OL_{n-1} - \sum_{s=0}^{7}(L_{s+0} + L_{s+1})\mathrm{e_s} + \sum_{s=2}^{n-3} OL_s. \qquad \square$$

Proposition 6.5. *The sum of the numbers with even indexes of Leonardo's octonions is given by:*

$$\sum_{m=2}^{n} OL_{2m} = 2OL_{2n-1} - \sum_{s=0}^{7} L_{s+1}\mathrm{e_s} + \sum_{s=1}^{n-2} OL_{2s+1}.$$

Proof. Using the recurrence relation of Leonardo's octonsion with $n \in \mathbb{N}$, we have:

$$OL_n = 2OL_{n-1} - OL_{n-3}.$$

Thus, evaluating the recurrence relation in values of $n \geq 3$, we get:

$$\begin{aligned} OL_4 &= 2OL_3 - OL_1, \\ OL_6 &= 2OL_5 - OL_3, \\ OL_8 &= 2OL_7 - OL_5, \\ &\vdots \\ OL_{2n-2} &= 2OL_{2n-3} - OL_{2n-5}, \\ OL_{2n} &= 2OL_{2n-1} - OL_{2n-3}. \end{aligned}$$

Through successive cancellations, we obtain:

$$\begin{aligned} \sum_{m=2}^{n} OL_{2m} &= OL_3 - OL_1 + OL_5 + \cdots + OL_{2n-3} + 2OL_{2n-1} \\ &= 2OL_{2n-1} - OL_1 + \sum_{s=3}^{2n-3} OL_s. \end{aligned}$$

Considering the initial values through Definition 4.1, it follows that:

$$2OL_{2n-1} - \sum_{s=0}^{7} L_{s+1}\mathrm{e_s} + \sum_{s=1}^{n-2} OL_{2s+1}. \qquad \square$$

Proposition 6.6. *The sum of the numbers with odd indices of Leonardo's octonions is given by:*

$$\sum_{m=2}^{n} OL_{2m-1} = 2OL_{2n-2} - \sum_{s=0}^{7} L_{s+0}\mathrm{e_s} + \sum_{s=0}^{n-3} OL_{2s+2}.$$

Proof. Using the recurrence relation of Leonardo's octonions with $n \in \mathbb{N}$, we have:

$$OL_n = 2OL_{n-1} - OL_{n-3}.$$

Thus, evaluating the recurrence relation, in values of $n \geq 3$, we get:

$$\begin{aligned} OL_3 &= 2OL_2 - OL_0, \\ OL_5 &= 2OL_4 - OL_2, \\ OL_7 &= 2OL_6 - OL_4, \\ &\vdots \\ OL_{2n-3} &= 2OL_{2n-4} - OL_{2n-6}, \\ OL_{2n-1} &= 2. \end{aligned}$$

Through successive cancellations, we obtain:

$$\begin{aligned} \sum_{m=2}^{n} OL_{2m-1} &= OL_2 - OL_0 + OL_4 + \cdots + OL_{2n-4} + 2OL_{2n-2} \\ &= 2OL_{2n-2} - OL_0 + \sum_{s=0}^{n-3} OL_{2s+1}. \end{aligned}$$

Considering the initial values through Definition 4.1, it follows that:

$$2OL_{2n-2} - \sum_{s=0}^{7} L_{s+0}\mathrm{e_s} + \sum_{s=2}^{2n-4} OL_s. \qquad \square$$

7. Conclusion

This work presents a discussion about the evolutionary process of Leonardo's sequence. When complexifying this sequence, it is possible to present the dimensional growth of the sequence from the insertion of the imaginary unit i, thus presenting Leonardo's Gaussians. And yet, it was possible to approach the terms of Leonardo's sequence in eight dimensions, obtaining Leonardo's octonions.

Moreover, the generating functions, Binet's formula, matrix forms, generalizations and properties linked to these numbers were also presented. Finally, this article makes it possible to contribute to the mathematical field and provides mathematical researchers with knowledge about Leonardo's sequence and its evolutionary process.

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