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SINE SUBTRACTION LAWS ON SEMIGROUPS

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Abstract. We consider two variants of the sine subtraction law on a semigroup S. The main objective is to solve $f(xy^*) = f(x)g(y) - g(x)f(y)$ for unknown functions $f, g\colon S \to \mathbb{C}$, where $x \mapsto x^*$ is an anti-homomorphic involution. Until now this equation was not solved even when S is a non-Abelian group and $x^* = x^{-1}$. We find the solutions assuming that f is central. A secondary objective is to solve $f(x\sigma(y)) = f(x)g(y) - g(x)f(y)$, where $\sigma\colon S \to S$ is a homomorphic involution. Until now this variant was solved assuming that S has an identity element. We also find the continuous solutions of these equations on topological semigroups.

1. Introduction

In this article S is a semigroup equipped with an involution that may be either homomorphic or anti-homomorphic. We consider two extensions of the sine subtraction law to semigroups. For the anti-homomorphic case $x \mapsto x^*$ denotes a self-mapping of S such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. Our main goal is to solve the functional equation

$$f(xy^*) = f(x)g(y) - g(x)f(y), \quad x, y \in S, \tag{1.1}$$

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for unknown functions $f, g\colon S \to \mathbb{C}$. On groups an obvious candidate for the involution is $x^* = x^{-1}$. We also discuss the variant

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad x, y \in S, \tag{1.2}$$

where $\sigma\colon S \to S$ is a homomorphic involution, that is $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in S$. We will refer to both (1.1) and (1.2) as *sine subtraction laws*, with the distinction evident from the notation used. Clearly they are equivalent if S is commutative.

Both equations were solved more than a century ago by Wilson [8] for the case that S is a 2-divisible Abelian group and the involution is the group inverse. Later the condition of 2-divisibility was dropped (see [1, pp. 216-217]). The general solution of the anti-homomorphic variant (1.1) is not known, even on non-Abelian groups with $x^* = x^{-1}$. The homomorphic variant (1.2) was solved in [4] for the case that S is a monoid generated by its squares, then in [2] for all monoids.

Section 2 includes some notation, terminology, and known results about the sine addition law on semigroups. In section 3 we solve (1.1) under the assumption that f is central. In section 4 we make a small improvement to the results about (1.2) by finding its solution on a general semigroup. The main results are Theorems 3.2 and 4.2. We also give an example showing that non-central solutions of (1.1) exist on non-Abelian groups. This contrasts with the results about (1.2) which show that all solutions are central. Some examples are given in section 5, and the final section includes the solutions of some systems of equations.

Note that $\mathbb{C}$ can be replaced as co-domain by any quadratically closed commutative field which is uniquely 2-divisible.

2. Notation, terminology, and preliminaries

For simplicity we shall use the term *involution* for both homomorphic and anti-homomorphic involutions. There is no confusion since we always use the notation σ for a homomorphic involution and $x \mapsto x^*$ for an anti-homomorphic involution.

For any function $F\colon S \to \mathbb{C}$, define $\check{F}\colon S \to \mathbb{C}$ by $\check{F}(x) := F(x^*)$ for all $x \in S$.

A function F on S is said to be *odd* (with respect to the involution) if $\check{F} = -F$, resp. $F \circ \sigma = -F$. Similarly F is *even* if $\check{F} = F$, resp. $F \circ \sigma = F$.

A function $A\colon S \to \mathbb{C}$ is *additive* if $A(xy) = A(x) + A(y)$ for all $x, y \in S$.

A function $m\colon S \to \mathbb{C}$ is *multiplicative* if $m(xy) = m(x)m(y)$ for all $x, y \in S$. If in addition $m \neq 0$ then we call m an *exponential*. For a multiplicative function $m\colon S \to \mathbb{C}$, define the subsets

$$I_m := \{x \in S \mid m(x) = 0\}, \quad \text{and}$$

$$P_m := \{p \in I_m \setminus I_m^2 \mid up, pv, upv \in I_m \setminus I_m^2 \text{ for all } u, v \in S \setminus I_m\},$$

where $T^2 := \{xy \mid x, y \in T\}$ for any set T.

A function F on S is said to be *central* if $F(xy) = F(yx)$ for all $x, y \in S$.

A *monoid* is a semigroup with an identity element. For a topological semigroup S, let $C(S)$ denote the algebra of continuous functions mapping S into $\mathbb{C}$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

It is not surprising that the *sine addition law*

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in S, \tag{2.1}$$

plays a key role in our work. The following is [3, Theorem 3.1].

Proposition 2.1. *Let S be a semigroup, and suppose $f, g\colon S \to \mathbb{C}$ satisfy the sine addition law* (2.1) *with $f \neq 0$. Then there exist multiplicative functions $m_1, m_2\colon S \to \mathbb{C}$ such that*

$$g = \frac{m_1 + m_2}{2}$$

and f has one of the following three forms.

(i) *For $m_1 \neq m_2$ there exists $c \in \mathbb{C}^*$ such that $f = c(m_1 - m_2)$.*

(ii) *For $m_1 = m_2 =: m \neq 0$ we have*

$$f(x) = \begin{cases} A(x)m(x) & \text{for } x \in S \setminus I_m, \\ 0 & \text{for } x \in I_m \setminus P_m, \\ f_P(x) & \text{for } x \in P_m, \end{cases} \tag{2.2}$$

where $A\colon S \setminus I_m \to \mathbb{C}$ is additive and f_P is the restriction of f to P_m. In addition we have

(a) *$f(sx) = f(xs) = 0$ for all $s \in I_m \setminus P_m$ and $x \in S \setminus I_m$; and*

(b) *if $x \in \{up, pv, upv\}$ for $p \in P_m$ and $u, v \in S \setminus I_m$, then $x \in P_m$ and we have respectively $f_P(x) = f_P(p)m(u)$, $f_P(x) = f_P(p)m(v)$, or $f_P(x) = f_P(p)m(uv)$.*

(iii) *For* $S \neq S^2$ *we have* $g = m_1 = m_2 = 0$ *and*

$$f(x) = \begin{cases} f_0(x) & \text{for } x \in S \setminus S^2, \\ 0 & \text{for } x \in S^2, \end{cases} \tag{2.3}$$

where $f_0 \colon S \setminus S^2 \to \mathbb{C}$ *(the restriction of* f *to* $S \setminus S^2$*) is an arbitrary nonzero function.*

Conversely the functions f, g *described above are solutions of* (2.1).

Furthermore, if S *is a topological semigroup and* $f \in C(S)$*, then* m_1, m_2, $m \in C(S)$.

Note that it is possible for some values of f_P to be chosen arbitrarily (see [3, Remark 3.2] for details). Note also that solution class (iii) does not arise on monoids, semigroups generated by their squares, or regular semigroups.

We introduce the following notation for convenience.

Notation 2.2. Let S be a semigroup and $m \colon S \to \mathbb{C}$ an exponential. The symbol ϕ_m shall denote a function of the form (2.2) as described in case (ii) of Proposition 2.1. Thus ϕ_m satisfies the special sine addition formula

$$\phi_m(xy) = \phi_m(x)m(y) + m(x)\phi_m(y), \quad x, y \in S.$$

As a reminder we sometimes write "ϕ_m has companion m".

Note that if $I_m = \emptyset$, for example if S is a group, then $\phi_m = Am$ for some additive $A \colon S \to \mathbb{C}$.

3. Sine subtraction law with an anti-homomorphic involution

In this section S is a semigroup with anti-homomorphic involution $x \mapsto x^*$.

Lemma 3.1. *Suppose* $f, g \colon S \to \mathbb{C}$ *satisfy* (1.1) *with* $\{f, g\}$ *linearly independent. Then* f *is odd.*

Proof. Computing $f(x(yz)^*)$ in two ways, we have

$$\begin{aligned} f(x(yz)^*) &= f(x)g(yz) - g(x)f(yz) \\ &= f(x)g(yz) - g(x)[f(y)g(z^*) - g(y)f(z^*)], \end{aligned}$$

and

$$f(x(yz)^*) = f(xz^*y^*) = f(xz^*)g(y) - g(xz^*)f(y)$$
$$= [f(x)g(z) - g(x)f(z)]g(y) - g(xz^*)f(y)$$

for all $x, y, z \in S$. Comparing these results we find that

$$\begin{aligned} f(x)[g(yz) - g(y)g(z)] + g(x)g(y)(f + \check{f})(z) \\ = [g(x)g(z^*) - g(xz^*)]f(y) \end{aligned} \tag{3.1}$$

for all $x, y, z \in S$. Taking $x = x_0$ such that $f(x_0) \neq 0$ (which exists by the independence assumption), we get

$$g(yz) - g(y)g(z) = f(y)h(z) + \lambda g(y)(f + \check{f})(z), \quad y, z \in S,$$

for some function $h\colon S \to \mathbb{C}$ and $\lambda \in \mathbb{C}$. Using this relation in (3.1), we get after some rearrangement that

$$f(x)f(y)(h + \check{h})(z)$$
$$= -\big(\lambda f(x)g(y) + g(x)g(y) + \lambda g(x)f(y)\big)(f + \check{f})(z).$$

Putting $x = y = x_0$ here we see that $h + \check{h} = \mu(f + \check{f})$ for some $\mu \in \mathbb{C}$, and with this the preceding equation can be written as

$$0 = \big(f(x)[\lambda g(y) + \mu f(y)] + g(x)[\lambda f(y) + g(y)]\big)(f + \check{f})(z),$$

for all $x, y, z \in S$. Thus by the linear independence of $\{f, g\}$ we see that $f + \check{f} = 0$. □

The following is the primary result of the paper.

Theorem 3.2. *Let S be a semigroup with involution $x \mapsto x^*$, and suppose $f, g\colon S \to \mathbb{C}$ are solutions of the sine subtraction law* (1.1) *with f central. Then f, g belong to one of the following families, where $m\colon S \to \mathbb{C}$ is an exponential, $b \in \mathbb{C}^*$, and $c \in \mathbb{C}$.*

(a) *$f = 0$ and g is arbitrary.*
(b) *$S \neq S^2$, $g = \lambda f$ for some $\lambda \in \mathbb{C}$, and f has the form* (2.3) *with arbitrary nonzero function $f_0\colon S \setminus S^2 \to \mathbb{C}$.*

(c) *For $m \neq \check{m}$ we have*

$$f = b(m - \check{m}), \qquad g = \frac{m + \check{m}}{2} + c\,\frac{m - \check{m}}{2}.$$

(d) *For $m = \check{m}$ and $\phi_m = -\check{\phi}_m \neq 0$, we have $f = \phi_m$ and $g = m + c\phi_m$.*

Conversely the pairs (f, g) so described are solutions of (1.1) *with f central.*

Furthermore if S is a topological semigroup and $f, g \in C(S)$, then $m, \check{m} \in C(S)$.

Proof. Clearly if $f = 0$ then g is arbitrary, and this is case (a). Henceforth we assume $f \neq 0$.

Next suppose $g = \lambda f$ for some $\lambda \in \mathbb{C}$. Then (1.1) reduces to $f(xy^*) = 0$ for all $x, y \in S$, so f vanishes everywhere on S^2. This contradicts $f \neq 0$ if $S = S^2$, so we must have $S \neq S^2$ and hence case (b). From now on we assume that g and f are linearly independent.

By Lemma 3.1 we see that f is odd. Replacing y by y^* in (1.1) we have $f(xy) = f(x)g(y^*) - g(x)f(y^*)$, thus

$$f(xy) = f(x)g(y^*) + g(x)f(y), \quad x, y \in S. \tag{3.2}$$

Since f is central this yields

$$f(y)g(x^*) + g(y)f(x) = f(yx) = f(xy) = f(x)g(y^*) + g(x)f(y),$$

so

$$f(y)[g(x^*) - g(x)] = f(x)[g(y^*) - g(y)], \quad x, y \in S.$$

Because $f \neq 0$ it follows that

$$\check{g} - g = \delta f$$

for some $\delta \in \mathbb{C}$. Using this relation in (3.2) we have

$$f(xy) = f(x)g(y) + g(x)f(y) + \delta f(x)f(y), \quad x, y \in S.$$

Defining $g' \colon S \to \mathbb{C}$ by

$$g' := g + \frac{\delta}{2} f, \tag{3.3}$$

the preceding equation becomes the sine addition formula

$$f(xy) = f(x)g'(y) + g'(x)f(y), \quad x, y \in S,$$

with solutions provided by Proposition 2.1. Note also that $g' \neq 0$ by the independence of $\{f, g\}$, so case (iii) of Proposition 2.1 is ruled out.

From Proposition 2.1 case (i) we get $f = b(m_1 - m_2)$ and $g' = (m_1 + m_2)/2$ for some $b \in \mathbb{C}^*$ and distinct multiplicative $m_1, m_2 \colon S \to \mathbb{C}$. Thus by (3.3) we have

$$g = \frac{m_1 + m_2}{2} + c\,\frac{m_1 - m_2}{2}$$

where $c := -\delta b$. Furthermore since f is odd and $b \neq 0$ we see that

$$\check{m}_1 - \check{m}_2 = -(m_1 - m_2).$$

By [5, Corollary 3.19], since $m_1 \neq m_2$ it follows that $\check{m}_2 = m_1$ and $\check{m}_1 = m_2$. Defining $m := m_1$ we have $m_2 = \check{m} \neq m$, and this is our family (c).

From Proposition 2.1 case (ii) we get $f = \phi_m$ and $g' = m$, where m is an exponential and ϕ_m has companion m. Since f is odd and nonzero we have $\check{\phi}_m = -\phi_m \neq 0$. Using the fact that the pair (ϕ_m, m) is a solution of (2.1) with ϕ_m odd, we find that this pair satisfies (1.1) if and only if

$$\begin{aligned} \phi_m(x)m(y) - m(x)\phi_m(y) &= \phi_m(xy^*) \\ &= \phi_m(x)m(y^*) + m(x)\phi_m(y^*) \\ &= \phi_m(x)m(y^*) - m(x)\phi_m(y) \end{aligned}$$

for all $x, y \in S$. Since $\phi_m \neq 0$ this implies $\check{m} = m$ and we have family (d).

The converses are straightforward verifications.

The topological statement follows from Proposition 2.1 and the construction. □

The following example (based on [6, Example 1]) shows that there exist non-central solutions of (1.1), even on non-Abelian groups with $x^* = x^{-1}$.

Example 3.3. Let G be the $(ax + b)$-group

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0,\ b \in \mathbb{R} \right\},$$

with involution $X^* = X^{-1}$ for all $X \in G$. Define $f, g \colon G \to \mathbb{C}$ by

$$f\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \frac{b}{\sqrt{a}}, \quad g\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \sqrt{a}, \quad \text{for all } \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G.$$

Then it is easy to check that (f, g) is a solution of (1.1) and f is not central.

As we will see in the next section, all solutions of (1.2) are central. This example shows that the same is not true of (1.1).

4. Sine subtraction law with a homomorphic involution

In this section S is a semigroup with a homomorphic involution σ. As noted earlier the general solution of (1.2) is known on monoids. Here we extend the results to a general semigroup. For any function $F\colon S \to \mathbb{C}$ let

$$F_e := \frac{F + F\circ\sigma}{2} \quad \text{and} \quad F_o := \frac{F - F\circ\sigma}{2}$$

denote the even and odd parts of F, respectively.

LEMMA 4.1. *Suppose $f, g\colon S \to \mathbb{C}$ satisfy (1.2) with $\{f, g\}$ linearly independent. Then f is odd and $g_o = cf$ for some constant $c \in \mathbb{C}$.*

PROOF. We start by calculating $f(x\sigma(yz))$ in two different ways. By (1.2) we have

$$\begin{aligned} f(x\sigma(yz)) &= f(x)g(yz) - g(x)f(yz) \\ &= f(x)g(yz) - g(x)[f(y)g(\sigma(z)) - g(y)f(\sigma(z))], \end{aligned}$$

and

$$\begin{aligned} f(x\sigma(y)\sigma(z)) &= f(x\sigma(y))g(z) - g(x\sigma(y))f(z) \\ &= [f(x)g(y) - g(x)f(y)]g(z) - g(x\sigma(y))f(z), \end{aligned}$$

for all $x, y, z \in S$. It follows that

$$\begin{aligned} (4.1)\quad & f(x)[g(yz) - g(y)g(z)] \\ &= -g(x)[f(y)g(z) - f(y)g\circ\sigma(z) + g(y)f\circ\sigma(z)] - g(x\sigma(y))f(z) \\ &= -g(x)[2f(y)g_o(z) + g(y)f\circ\sigma(z)] - g(x\sigma(y))f(z) \end{aligned}$$

for all $x, y, z \in S$. Choosing $x = x_0$ such that $f(x_0) \neq 0$ (which exists by linear independence), we find that

$$(4.2)\qquad g(yz) - g(y)g(z) = \alpha[2f(y)g_o(z) + g(y)f \circ \sigma(z)] + h(y)f(z), \quad y, z \in S,$$

for some $\alpha \in \mathbb{C}$ and $h\colon S \to \mathbb{C}$. Using (4.2) in (4.1) we get after some rearrangement

$$(4.3)\quad (g + \alpha f)(x)[2f(y)g_o(z) + g(y)f \circ \sigma(z)] = -[g(x\sigma(y)) + f(x)h(y)]f(z).$$

Putting $z = x_0$ here we get

$$g(x\sigma(y)) + f(x)h(y) = (g + \alpha f)(x)(\beta_1 f + \beta_2 g)(y), \quad x, y \in S,$$

for some $\beta_1, \beta_2 \in \mathbb{C}$. Putting this back into (4.3) we arrive at

$$(g + \alpha f)(x)[2f(y)g_o(z) + g(y)f \circ \sigma(z) + (\beta_1 f + \beta_2 g)(y)f(z)] = 0.$$

By the linearly independence of $\{f, g\}$ we can choose $x = x_1$ such that $(g + \alpha f)(x_1) \neq 0$, and the preceding equation simplifies to

$$f(y)[2g_o(z) + \beta_1 f(z)] + g(y)[f \circ \sigma(z) + \beta_2 f(z)] = 0, \quad y, z \in S.$$

Applying the independence to this equation we see that $g_o = cf$ for some $c \in \mathbb{C}$, and

$$(4.4)\qquad f \circ \sigma = -\beta_2 f.$$

It only remains to show that f is odd. By (4.4) we have

$$f = f \circ \sigma \circ \sigma = -\beta_2 f \circ \sigma = \beta_2^2 f,$$

so $\beta_2 = \pm 1$. If $\beta_2 = 1$ then f is odd and we are finished.

To complete the proof we show that $\beta_2 = -1$ is impossible. If $\beta_2 = -1$ then (4.4) shows that f is even. Since f is even and $g_o = cf$ we see that g is also even. Now (1.2) with y replaced by $\sigma(y)$ becomes

$$(4.5)\qquad f(xy) = f(x)g(y) - g(x)f(y), \quad x, y \in S.$$

This shows that $f(xy) = -f(yx)$, therefore

$$f(x(yz)) = -f((yz)x) = -f(y(zx)) = f((zx)y) = f(z(xy)) = -f(xyz),$$

so $f(xyz) = 0$ for all $x, y, z \in S$. Using this together with (4.5) and (4.2) we get

$$\begin{aligned} 0 &= f(x(yz)) = f(x)g(yz) - g(x)f(yz) \\ &= f(x)[g(y)g(z) + \alpha g(y)f(z) + h(y)f(z)] - g(x)[f(y)g(z) - g(y)f(z)] \end{aligned}$$

for all $x, y, z \in S$. This contradicts the linear independence of $\{f, g\}$. □

Now we come to the secondary main result, which generalizes [2, Corollary 4.3] by dropping the requirement that S contains an identity element.

THEOREM 4.2. *Let S be a semigroup with involution σ, and suppose $f, g\colon S \to \mathbb{C}$ are solutions of the sine subtraction law* (1.2). *Then f, g belong to one of the following families, where $m\colon S \to \mathbb{C}$ is an exponential, $b \in \mathbb{C}^*$, and $c \in \mathbb{C}$.*

(a) *$f = 0$ and g is arbitrary.*

(b) *$S \neq S^2$, $g = \lambda f$ for some $\lambda \in \mathbb{C}$, and f is given by* (2.3) *with arbitrary nonzero function $f_0\colon S \setminus S^2 \to \mathbb{C}$.*

(c) *For $m \neq m \circ \sigma$ we have*

$$f = b(m - m \circ \sigma), \qquad g = \frac{m + m \circ \sigma}{2} + c\,\frac{m - m \circ \sigma}{2}.$$

(d) *For $m = m \circ \sigma$ and $\phi_m = -\phi_m \circ \sigma \neq 0$, we have $f = \phi_m$ and $g = m + c\phi_m$.*

Conversely the pairs (f, g) so described are solutions of (1.2).

Moreover, if S is a topological semigroup and $f, g \in C(S)$, then $m, m \circ \sigma \in C(S)$.

PROOF. The proof is accomplished by using Lemma 4.1 to replace part of the proof of [2, Corollary 4.3]. That proof is an adaptation of [5, Theorem 4.12], which gave the solutions of (1.2) on groups.

If $f = 0$ then g is arbitrary and we have solution (a). Henceforth we assume $f \neq 0$.

Next suppose $g = \lambda f$ for some $\lambda \in \mathbb{C}$. Then (1.2) yields $f(x\sigma(y)) = 0$ for all $x, y \in S$. Thus f vanishes everywhere on S^2. Since $f \neq 0$ we have $S \neq S^2$ and the restriction of f to $S \setminus S^2$ is a nonzero function, so we are in case (b).

It remains to treat the case that $\{f, g\}$ is linearly independent. By Lemma 4.1 we see that f is odd and $g_o = cf$ for some $c \in \mathbb{C}$. Since the existence of an identity element in S was needed in the proof of [5, Theorem 4.12] only to show that f is odd and $g_o = cf$, the rest follows almost unchanged from that proof. You just substitute Proposition 2.1 for [5, Corollary 4.4] when the solutions of the sine addition formula are needed. That causes a modification in family (d), where we replace the form $f = Am$ (with A additive and m exponential) by the form $f = \phi_m$ as described in Proposition 2.1 part (ii). □

5. Examples

The examples in this section are for semigroups with anti-homomorphic involutions $x \mapsto x^*$. (Examples for homomorphic involutions can be found in [4, 2].) For each semigroup we identify the forms of exponential m and the corresponding ϕ_m to be used in Theorem 3.2.

As noted earlier, an exponential m on a group never takes the value 0, so $I_m = \emptyset$ and we have $\phi_m = Am$ for some additive function A on the group. Also $S = S^2$ for every monoid, including groups.

Let $M(n, \mathbb{C})$ denote the monoid of $n \times n$ complex matrices under multiplication and the usual topology, and $GL(n, \mathbb{C})$ its subgroup of invertible elements. Let $\Re(\alpha)$ be the real part of a complex number α. For the matrix groups in our first three examples we take $X^* := X^{-1}$ for all X.

Example 5.1. Let G be the general linear group $GL(n, \mathbb{C})$. By [4, Lemma 5.4] the continuous exponentials $m \in C(G)$ have the form

$$m_{n,\lambda}(X) = |\det(X)|^{\lambda - n}(\det(X))^n, \quad X \in GL(n, \mathbb{C}),$$

where $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, and the continuous additive functions $A \in C(G)$ are given by

$$A_\delta(X) = \delta \log|\det(X)|, \quad X \in GL(n, \mathbb{C}), \tag{5.1}$$

where $\delta \in \mathbb{C}$.

An exponential $m_{n,\lambda}$ is even if and only if $m_{n,\lambda} = m_{0,0} = 1$. For $m = 1$ we find that the corresponding $\phi_m = A_\delta$ is odd for every $\delta \in \mathbb{C}$.

The second example is the group of affine motions of $\mathbb{R}$.

Example 5.2. Consider the $(ax+b)$-group G defined in Example 3.3, with the topology inherited from $M(2, \mathbb{C})$. By [5, Example 3.13] the continuous exponentials $m \in C(G)$ have the form

$$m_\lambda \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a^\lambda, \qquad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G,$$

where $\lambda \in \mathbb{C}$, and by [5, Example 2.10] the continuous additive functions $A \in C(G)$ are of the form

$$A_\gamma \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \gamma a, \qquad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G,$$

where $\gamma \in \mathbb{C}$.

An exponential m_λ is even if and only

$$a^{-\lambda} = m_\lambda \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} = m_\lambda \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a^\lambda$$

for all $a > 0$, so $\lambda = 0$ and thus $m_\lambda = m_0 = 1$. For this exponential the corresponding $\phi_m = A_\gamma$ is odd if and only if

$$\gamma a^{-1} = A_\gamma \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} = -A_\gamma \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = -\gamma a$$

for all $a > 0$, so $\gamma = 0$ and therefore $\phi_m = 0$.

The third example is the *Heisenberg group*

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

EXAMPLE 5.3. Let $G = H_3$ with the topology inherited from $M(3, \mathbb{C})$. By [5, Example 3.14] the continuous exponentials $m \in C(G)$ have the form

$$m_{\alpha,\beta} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = e^{\alpha x + \beta y}, \quad x, y, z \in \mathbb{R},$$

and by [5, Example 2.11] the continuous additive functions $A \in C(G)$ are given by

$$A_{\lambda,\mu} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \lambda x + \mu y, \quad x, y, z \in \mathbb{R},$$

where $\alpha, \beta, \lambda, \mu \in \mathbb{C}$.

An exponential $m_{\alpha,\beta}$ is even if and only

$$e^{-\alpha x-\beta y} = m_{\alpha,\beta}\begin{pmatrix}1 & -x & -z+xy\\ 0 & 1 & -y\\ 0 & 0 & 1\end{pmatrix} = m_{\alpha,\beta}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = e^{\alpha x+\beta y}$$

for all $x, y, z \in \mathbb{R}$, so $\alpha = \beta = 0$. For an even exponential $m = m_{0,0} = 1$ we find that $\phi_m = A_{\lambda,\mu}$ is odd if and only if

$$-\lambda x - \mu y = A_{\lambda,\mu}\begin{pmatrix}1 & -x & -z+xy\\ 0 & 1 & -y\\ 0 & 0 & 1\end{pmatrix} = -A_{\lambda,\mu}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = -(\lambda x + \mu y)$$

for all $x, y, z \in \mathbb{R}$. This is true for every $\lambda, \mu \in \mathbb{C}$.

For the next example let $X^* = \overline{X}^t$ be the adjoint matrix of $X \in M(2,\mathbb{C})$. Note that the non-commutative monoid $S = M(2,\mathbb{C})$ is generated by its squares (see [4, p. 192]). We claim that P_m is empty for any multiplicative $m\colon S \to \mathbb{C}$. Indeed, given any $X \in I_m$ there exist $n \in \mathbb{N}$ and $Y_1, \dots, Y_n \in S$ such that $X = Y_1^2 \cdots Y_n^2$. Hence $0 = m(X) = m(Y_1^2 \cdots Y_n^2) = m(Y_1)^2 \cdots m(Y_n)^2$, so $m(Y_j) = 0$ for some j. It follows that $X = (Y_1^2 \cdots Y_{j-1}^2 Y_j)\cdot(Y_j Y_{j+1}^2 \cdots Y_n^2) \in I_m^2$, therefore $I_m = I_m^2$ and so $P_m = \emptyset$.

Example 5.4. Let $S = M(2,\mathbb{C})$. By [4, Lemma 5.4] the continuous exponentials $m \in C(S)$ have the form $m = 1$ or

$$m(X) = \begin{cases} |\det(X)|^{\lambda-n}(\det(X))^n & \text{if } \det(X) \neq 0,\\ 0 & \text{if } \det(X) = 0, \end{cases} \tag{5.2}$$

where $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ with $\mathfrak{R}(\lambda) > 0$. As we saw in Example 5.1 the continuous additive functions on $GL(2,\mathbb{C})$ have the form (5.1), and all of them are odd. The only additive function on the whole monoid S is $A = 0$, since S contains a zero.

The only ϕ_m with companion $m = 1$ is the zero function, since $\phi_m = A$ is additive on S. If $m \in C(S)$ is given by (5.2), then (since $P_m = \emptyset$) the form of ϕ_m with companion m simplifies to

$$\phi_m(X) = \begin{cases} A(X)m(X) & \text{if } \det(X) \neq 0,\\ 0 & \text{if } \det(X) = 0, \end{cases} \tag{5.3}$$

where A has the form (5.1).

The exponential $m = 1$ is clearly even. An exponential m of the form (5.2) is even if and only $n = 0$. For $m = 1$ we see that $\phi_m = 0$ is odd. If an m of the form (5.2) is even, then by (5.1), (5.3), and (5.2) with $n = 0$, we see that ϕ_m of the form (5.3) is odd if and only if

$$\delta \log|\det(X)||\det(X)|^{\lambda} = -\delta \log|\det(X^*)||\det(X^*)|^{\lambda}, \quad \text{for all } X \in GL(n, \mathbb{C}).$$

Thus $\delta = 0$ and we have $\phi_m = 0$.

The final example is a semigroup for which $S \neq S^2$.

Example 5.5. Let $S = \mathbb{N} \times \mathbb{N}$ equipped with (vector) addition and the discrete topology. Define

$$(x, y)^* := (y, x), \quad \text{for all } (x, y) \in S.$$

By [7, Example 8.3] the exponentials $m \colon S \to \mathbb{C}$ have the form

$$m_{b_1,b_2}(x, y) = b_1^x b_2^y, \quad \text{for all } x, y \in \mathbb{N},$$

for some $b_1, b_2 \in \mathbb{C}^*$, and the additive functions $A \colon S \to \mathbb{C}$ are given by

$$A_{c_1,c_2}(x, y) = c_1 x + c_2 y, \quad \text{for all } x, y \in \mathbb{N},$$

where $c_1, c_2 \in \mathbb{C}$.

Clearly an exponential $m = m_{b_1,b_2}$ never takes the value zero at any point of S, so $\phi_m = A_{c_1,c_2} m$ for some additive $c_1, c_2 \in \mathbb{C}$.

An exponential m_{b_1,b_2} is even if and only if $b_1^x b_2^y = b_1^y b_2^x$ for all $x, y \in \mathbb{N}$, which implies that $b_1 = b_2$. For an even $m = m_{b,b}$, the corresponding $\phi_m = A_{c_1,c_2} m$ is odd if and only if $c_1 x + c_2 y = -(c_1 y + c_2 x)$ for all $x, y \in \mathbb{N}$. Thus $c_2 = -c_1$, so $\phi_m = A_{c,-c} m$ for some $c \in \mathbb{C}$.

Since $S \neq S^2$ it remains to describe the form of f given by (2.3) in Proposition 2.1(iii). Here

$$S^2 = \{(x_1, y_1) + (x_2, y_2) \mid x_1, x_2, y_1, y_2 \in \mathbb{N}\} = \{(m, n) \in S \mid m \geq 2, n \geq 2\},$$

so $S \setminus S^2 = (\{1\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1\})$ and f_0 is an arbitrary nonzero function on this set.

6. Systems of equations

We close the paper by giving the general solutions of systems of equations combining each pair of (1.1), (2.1), and the *cosine addition law*

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in S. \tag{6.1}$$

For the latter equation we have the following, which is [2, Theorem 3.2] supplemented with further details in family (iii) as recorded in Proposition 2.1 case (ii).

Proposition 6.1. *Let S be a semigroup. The solutions $g, f\colon S \to \mathbb{C}$ of the cosine addition law* (6.1) *are the following families, where $m, m_1, m_2\colon S \to \mathbb{C}$ are multiplicative functions with $m \neq 0$ and $m_1 \neq m_2$.*

(a) $g = f = 0$.

(b) $S \neq S^2$, $g = \pm f$, *and f has the form* (2.3) *with arbitrary nonzero function* $f_0\colon S \setminus S^2 \to \mathbb{C}$.

(c) $g = \dfrac{c^{-1}m_1 + cm_2}{c^{-1} + c}$ *and* $f = \dfrac{m_1 - m_2}{i(c^{-1} + c)}$, *where* $c \in \mathbb{C}^* \setminus \{\pm i\}$.

(d) $g = m \pm \phi_m$ *and* $f = \phi_m$ *with companion* m.

Furthermore, if S is a topological semigroup and $g, f \in C(S)$, then $m, m_1, m_2 \in C(S)$.

For the systems below we need not assume any form of commutativity (i.e. that any function is central), since all solutions of either addition law (2.1) or (6.1) are central. While the results are stated for the case of an anti-homomorphic involution, the obvious parallels hold for a homomorphic involution.

The first system consists of the sine addition and subtraction laws.

Corollary 6.2. *Let S be a semigroup with involution $x \mapsto x^*$, and suppose $f, g\colon S \to \mathbb{C}$ satisfy the system of equations* (1.1), (2.1). *The solutions are the following families, where $m\colon S \to \mathbb{C}$ is an exponential.*

(a) $f = 0$ *and g is arbitrary.*

(b) $S \neq S^2$, $g = 0$, *and f has the form* (2.3) *where $f_0\colon S \setminus S^2 \to \mathbb{C}$ is an arbitrary nonzero function.*

(c) *For $m \neq \check{m}$ and $b \in \mathbb{C}^*$,*

$$f = b(m - \check{m}), \qquad g = \frac{m + \check{m}}{2}.$$

(d) *For* $m = \check{m}$ *and* $\check{\phi}_m = -\phi_m \neq 0$, *we have* $f = \phi_m$ *and* $g = m$.

Moreover, if S *is a topological semigroup and* $f \in C(S)$, *then* $m, \check{m} \in C(S)$.

Proof. If $f = 0$ then g is arbitrary, and this is our family (a).

If $f \neq 0$ then f is central by Proposition 2.1, so we can apply Theorem 3.2. It is straightforward to check that cases (b), (c), (d) of Theorem 3.2 form solutions of (2.1) if and only if $\lambda = 0$, respectively $c = 0$. Thus we have the present families (b), (c), and (d).

The converse is evident, and the topological statement follows from Proposition 2.1. □

The next system pairs the sine subtraction law with the cosine addition law.

Corollary 6.3. *Let* S *be a semigroup with involution* $x \mapsto x^*$, *and suppose* $f, g\colon S \to \mathbb{C}$ *satisfy the system of equations* (1.1), (6.1). *The solutions are the following families, where* $m\colon S \to \mathbb{C}$ *is multiplicative.*

(a) $f = 0$ *and* $g = m$.

(b) $S \neq S^2$, $g = \pm f$, *and* f *has the form* (2.3) *with arbitrary nonzero function* $f_0\colon S \setminus S^2 \to \mathbb{C}$.

(c) *For* $m \neq \check{m}$ *and* $b \in \mathbb{C}^*$,

$$f = b(m - \check{m}), \qquad g = \frac{m + \check{m}}{2} \pm \sqrt{1 + 4b^2}\,\frac{m - \check{m}}{2}.$$

(d) *For* $m = \check{m} \neq 0$ *and* $\check{\phi}_m = -\phi_m \neq 0$, *we have* $f = \phi_m$ *and* $g = m \pm \phi_m$.

Moreover, if S *is a topological semigroup and* $f, g \in C(S)$, *then* $m, \check{m} \in C(S)$.

Proof. By Proposition 6.1 we again have f central, so Theorem 3.2 applies. If $f = 0$ then (6.1) shows that g is multiplicative and we have family (a).

Henceforth we assume $f \neq 0$. Case (b) of Theorem 3.2 yields a solution of (6.1) only if $\lambda = \pm 1$, and this is our family (b).

Substituting the formulas from case (c) of Theorem 3.2 into (6.1), we find after simplification that

$$0 = (1 - c^2 + 4b^2)[m(x) - \check{m}(x)][m(y) - \check{m}(y)], \quad x, y \in S.$$

Since $m \neq \check{m}$ we have $c^2 = 1 + 4b^2$, therefore we are in solution family (c).

The remaining case (d) of Theorem 3.2 is $f = \phi_m$ and $g = m + c\phi_m$ for some exponential $m \neq \check{m}$, nonzero $\phi_m = -\check{\phi}_m$, and $c \in \mathbb{C}$. This pair of functions satisfies (6.1) only if $c = \pm 1$, and this is our family (d).

Again the converse is clear, and the topological statement follows from Theorem 3.2. □

The combination of the sine and cosine addition laws is the strongest of the three systems in the sense that it has the shortest list of solution types.

Corollary 6.4. *Let S be a semigroup. The functions $f, g\colon S \to \mathbb{C}$ satisfy the system of equations* (2.1), (6.1) *if and only if there exist multiplicative functions $m_1, m_2\colon S \to \mathbb{C}$ such that*

$$f = \frac{m_1 - m_2}{2i}, \qquad g = \frac{m_1 + m_2}{2}. \tag{6.2}$$

Furthermore if S is a topological semigroup and $f \in C(S)$, then $m_1, m_2 \in C(S)$.

Proof. If $f = 0$, then f, g satisfy (6.1) only if g is multiplicative. This is (6.2) with $m_1 = m_2$.

If $f \neq 0$ then we apply Proposition 2.1 and check its solution families in (6.1). A small calculation reveals that the pair $f = b(m_1 - m_2)$ and $g = (m_1 + m_2)/2$, with $m_1 \neq m_2$, satisfies (6.1) only if $b^2 = -1/4$, thus we have (6.2) with $m_1 \neq m_2$. Case (ii) of Proposition 2.1, namely $g = m$ and $f = \phi_m$, satisfies (6.1) only if $\phi_m = 0$, contradicting $f \neq 0$. Finally case (iii) of Proposition 2.1 is not a solution of (6.1), since $g = 0$ in (6.1) implies $f = 0$, again contradicting $f \neq 0$.

The topological statement follows from Proposition 2.1. □

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