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EXISTENCE, DATA DEPENDENCE AND STABILITY OF FIXED POINTS OF MULTIVALUED MAPS IN INCOMPLETE METRIC SPACES

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Abstract. In this paper we formulate a setvalued fixed point problem by combining four prevalent trends of fixed point theory. We solve the problem by showing that the set of fixed points is nonempty. Further we have a data dependence result pertaining to the problem and also a stability result for the fixed point sets. The main result is extended to metric spaces with a graph. The results are obtained without the use of metric completeness assumption which is replaced by some other conditions suitable for solving the fixed point problem. There are some consequences of the main result. The main result is illustrated with an example.

1. Introduction and mathematical preliminaries

The development of fixed point theory of contractive mappings following the work of Banach has been very extensive and is carried into the recent times even after about hundred years of its initiation. Works like [6, 13, 17, 20, 22, 33, 35] are some instances from this line of research. A very influential form of contraction was proposed by Suzuki ([35]) who generalized the Banach contraction and in the sequel initiated a new trend in fixed point theory. Such

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mappings defined in line with the idea in [35] came to be known as Suzuki type contractions ([2, 16, 29]). Nadler ([28]) extended fixed point theory to the domain of setvalued analysis with the help of the idea of Hausdorff distance. Following the work of Nadler, fixed point studies of setvalued mappings have flourished in a large way. Comprehensive accounts of this development are obtainable in [3, 7, 24].

Again rational contractive inequalities occupy a prominent position in fixed point theory. It was initiated by Dass et al ([14]). The use of rational terms in contractive inequalities has been done in works like [4, 8, 21].

The use of admissibility conditions has come up prominently in fixed point theory. These are certain conditions on the behaviour of the contractive mapping under consideration and are brought about through a prescribed function. The advantage of using such conditions is that the contraction condition can be restricted to certain suitable pairs of points in which case there is no need to define contraction condition on the whole space. Recently, fixed point results using admissibility conditions have been developed in several works like [12, 19, 33].

The above trends of research have individually contributed very substantially to the development of fixed point theory. There are large scopes of putting these ideas together in order to create new results in fixed point theory. Accordingly we combine the above four existing trends to formulate a fixed point problem in metric spaces. We do not assume completeness property of the metric space. Rather we use an alternative condition on the metric space which is brought about through a separate function. We establish existence, data dependence and stability results relating to the fixed point problem formulated here. We extend our result to the case of a metric space with a graphic structure. Some of our results are illustrated with examples.

A data dependence problem is to estimate the distance between the fixed point sets of two operators when the functional value of these mappings at every point differs by a magnitude less than a given positive number. As multivalued mappings often have larger fixed point sets than their singlevalued counterparts, the study of data dependence problem within the domain of setvalued analysis assumes additional importance. It has important applications to differential and integral equations ([9, 32]). Several research papers on data dependence have been published in recent literature of which we mention a few in references [10, 12, 18].

Stability is a concept in dynamical systems related to limiting behaviors. There are various notions of stability both in discrete and continuous dynamical systems ([31]). In this article, stability is related limiting behaviour of the fixed point sets associated with a sequence of multivalued mappings to that of the limit function to which the sequence converges. There are several results dealing with the stability of fixed point sets as for instance the works noted in the references [5, 11, 12, 34].

In the following we give the technical details required for deduction of our results in the following sections.

Let (M,ρ) be a metric space and $CLB(M)$ be the class of all non-empty closed and bounded subsets of M. Define

$$\mathfrak{D}(a,B)=\inf\{\rho(a,b):b\in B\},\quad \text{where } a\in M \text{ and } B\in CLB(M),$$

$$\mathfrak{D}(A,B)=\inf\{\rho(a,b):a\in A,\ b\in B\},\quad \text{where } A,B\in CLB(M),$$

$$\mathcal{H}(A,B)=\max\{\sup_{x\in A}\mathfrak{D}(x,B),\sup_{y\in B}\mathfrak{D}(y,A)\},\quad \text{where } A,B\in CLB(M).$$

$\mathcal{H}$ is a metric on $CLB(M)$ and is called the Hausdorff–Pompeiu metric on $CLB(M)$. Moreover, if (M,ρ) is complete then $(CLB(M),\mathcal{H})$ is also complete ([28]).

Lemma 1.1 ([28]). *Let $A,B\in CLB(M)$ and $q>1$. Then for every $x\in A$ there exists $y\in B$ satisfying $\rho(x,y)\le q\,\mathcal{H}(A,B)$.*

Definition 1.1 ([28]). A point $u\in M$ is called a *fixed point* of a multivalued mapping $T\colon M\to CLB(M)$ if $u\in Tu$.

The fixed point set of T is denoted by F_T.

Definition 1.2 ([11]). A multivalued mapping $T\colon M\to CLB(M)$ is called *continuous at* $x\in M$ if $\mathcal{H}(Tx_n,Tx)\to 0$ as $n\to\infty$ for any sequence $\{x_n\}$ in M with $x_n\to x$ as $n\to\infty$.

The following ideas involve a function $\alpha\colon M\times M\to[0,\infty)$. The idea of the α-continuity of multivalued mappings has been introduced recently by Kutbi and Sintunavarat ([25]).

Definition 1.3 ([25]). A multivalued mapping $T\colon M\to CLB(M)$ is called *α-continuous at* $x\in M$ if $\lim_{n\to\infty}\mathcal{H}(Tx_n,Tx)=0$, whenever $\{x_n\}$ is a sequence in M with $x_n\to x$ as $n\to\infty$ and $\alpha(x_n,x_{n+1})\ge 1$ for all $n>0$.

Remark 1.1 ([25]). The continuity of a mapping guarantees its α-continuity but the converse may not be true.

Recently, the idea of α-completeness of a metric space has been introduced by Hussain et al ([19]).

Definition 1.4 ([19]). The metric space M is called *α-complete* if every Cauchy sequence $\{x_n\}$ in M satisfying $\alpha(x_n,x_{n+1})\ge 1$ for all $n>0$ is convergent in M.

Remark 1.2 ([19]). The completeness of a metric space M guarantees its α-completeness but the converse is not true.

Definition 1.5 ([12]). We say that a metric space M has *α-regular property* if $\alpha(x_n, x) \geq 1$ for all $n > 0$ whenever $\{x_n\}$ is a convergent sequence in M having limit $x \in M$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n > 0$.

Definition 1.6 ([11]). A multivalued mapping $T\colon M \to CLB(M)$ is called *α-admissible* if $\alpha(x, y) \geq 1$, for $x, y \in M$ implies $\alpha(u, v) \geq 1$, where $u \in Tx$ and $v \in Ty$.

In the following we define a multivalued contraction of Suzuki-type which unifies and generalizes many Suzuki type contractions in the existing literature [23, 27, 30, 35].

Definition 1.7. A multivalued mapping $T\colon M \to CLB(M)$ is said to be a *Suzuki-type α-contraction* if for $u, v \in M$ with $\alpha(u, v) \geq 1$,

$$\frac{1}{2}\mathfrak{D}(u, Tu) \leq \rho(u, v) \quad \text{implies} \quad \mathcal{H}(Tu, Tv) \leq qQ(u, v),$$

where

$$Q(u, v) = \max\left\{\rho(u, v), \mathfrak{D}(u, Tu), \mathfrak{D}(v, Tv), \frac{1}{2}[\mathfrak{D}(u, Tv) + \mathfrak{D}(v, Tu)], \sqrt{q}\frac{\mathfrak{D}(u, Tu)\mathfrak{D}(v, Tv)}{p + \mathcal{H}(Tu, Tv)}, \sqrt{q}\frac{\mathfrak{D}(u, Tv)\mathfrak{D}(v, Tu)}{r + \mathcal{H}(Tu, Tv)}\right\}$$

and $q \in (0, 1)$, $p, r > 0$.

2. Existence of nonempty fixed point set

Theorem 2.1. *Let (M, ρ) be a metric space and $\alpha\colon M \times M \to [0, \infty)$ be a mapping such that M is α-complete and has α-regular property. Let $T\colon M \to CLB(M)$ be such that* (i) *T is α-admissible,* (ii) *there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,* (iii) *T is a Suzuki-type α-contraction. Then F_T is nonempty.*

PROOF. By assumption (ii), there exists $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Since $q \in (0,1)$, we have $\frac{1}{\sqrt{q}} > 1$. As $Tx_0, Tx_1 \in CLB(M)$ and $x_1 \in Tx_0$, by Lemma 1.1, we find $x_2 \in Tx_1$ such that

$$\rho(x_1, x_2) \leq \frac{1}{\sqrt{q}} \mathcal{H}(Tx_0, Tx_1).$$

As $\alpha(x_0, x_1) \geq 1$, by assumption (i), we have $\alpha(x_1, x_2) \geq 1$. As $Tx_1, Tx_2 \in CLB(M)$, $x_2 \in Tx_1$ and $\frac{1}{\sqrt{q}} > 1$, there exists $x_3 \in Tx_2$ such that

$$\rho(x_2, x_3) \leq \frac{1}{\sqrt{q}} \mathcal{H}(Tx_1, Tx_2).$$

As $\alpha(x_1, x_2) \geq 1$, by assumption (i), we have $\alpha(x_2, x_3) \geq 1$. Arguing in this way we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} \in Tx_n, \quad \text{for all } n \geq 0, \tag{2.1}$$

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \geq 0, \tag{2.2}$$

$$\rho(x_{n+1}, x_{n+2}) \leq \frac{1}{\sqrt{q}} \mathcal{H}(Tx_n, Tx_{n+1}), \quad \text{for all } n \geq 0. \tag{2.3}$$

Now,

$$\frac{1}{2}\mathfrak{D}(x_n, Tx_n) \leq \frac{1}{2}\rho(x_n, x_{n+1}) \leq \rho(x_n, x_{n+1}), \quad \text{for all } n \geq 0. \tag{2.4}$$

Let

$$R_n = \rho(x_n, x_{n+1}), \quad \text{for all } n \geq 0.$$

By (2.2), (2.3) and (2.4), we have

$$\begin{aligned} \rho(x_{n+1}, x_{n+2}) &\leq \frac{1}{\sqrt{q}} \mathcal{H}(Tx_n, Tx_{n+1}) \\ &\leq \frac{1}{\sqrt{q}} q\, Q(x_n, x_{n+1}) \\ &= \sqrt{q}\, Q(x_n, x_{n+1}). \end{aligned} \tag{2.5}$$

Now,

$$
\begin{aligned}
Q(x_n, x_{n+1}) = \max\Big\{&\rho(x_n, x_{n+1}), \mathfrak{D}(x_n, Tx_n), \mathfrak{D}(x_{n+1}, Tx_{n+1}),\\
&\frac{1}{2}[\mathfrak{D}(x_n, Tx_{n+1}) + \mathfrak{D}(x_{n+1}, Tx_n)],\\
&\sqrt{q}\frac{\mathfrak{D}(x_n, Tx_n)\mathfrak{D}(x_{n+1}, Tx_{n+1})}{p + \mathcal{H}(Tx_n, Tx_{n+1})},\\
&\sqrt{q}\frac{\mathfrak{D}(x_n, Tx_{n+1})\mathfrak{D}(x_{n+1}, Tx_n)}{r + \mathcal{H}(Tx_n, Tx_{n+1})}\Big\}\\
\le \max\Big\{&\rho(x_n, x_{n+1}), \rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2}),\\
&\frac{1}{2}[\rho(x_n, x_{n+2}) + \rho(x_{n+1}, x_{n+1})],\\
&\sqrt{q}\frac{\rho(x_n, x_{n+1})\rho(x_{n+1}, x_{n+2})}{p + \sqrt{q}\rho(x_{n+1}, x_{n+2})}, \sqrt{q}\frac{\rho(x_n, x_{n+2})\rho(x_{n+1}, x_{n+1})}{r + \sqrt{q}\rho(x_{n+1}, x_{n+2})}\Big\}\\
\le \max\Big\{&\rho(x_n, x_{n+1}), \rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2}),\\
&\frac{1}{2}[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2})], \frac{\rho(x_n, x_{n+1})\rho(x_{n+1}, x_{n+2})}{\frac{p}{\sqrt{q}} + \rho(x_{n+1}, x_{n+2})}, 0\Big\}\\
\le \max\Big\{&\rho(x_n, x_{n+1}), \rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2}),\\
&\frac{1}{2}[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2})], \rho(x_n, x_{n+1}), 0\Big\}\\
= \max\Big\{&R_n, R_n, R_{n+1}, \frac{1}{2}[R_n + R_{n+1}], R_n, 0\Big\}\\
= \max\{&R_n, R_{n+1}\}, \ \Big[\text{ since } \frac{1}{2}[R_n + R_{n+1}] \le \max\{R_n, R_{n+1}\}\Big]. \qquad (2.6)
\end{aligned}
$$

If possible, suppose that $R_{n+1} > R_n \ge 0$. From (2.5) and the above inequality, we have

$$R_{n+1} \le \sqrt{q}\max\{R_n, R_{n+1}\} = \sqrt{q}\, R_{n+1} < R_{n+1},$$

which is a contradiction. Therefore, we have

$$R_{n+1} \le R_n, \quad \text{for all } n. \qquad (2.7)$$

From (2.5), (2.6) and (2.7), we get

$$R_{n+1} \leq \sqrt{q}\max\{R_n, R_{n+1}\} = \sqrt{q}\, R_n.$$

Applying the above inequality repeatedly, we have

$$R_{n+1} \leq \sqrt{q}\, R_n \leq (\sqrt{q})^2 R_{n-1} \leq (\sqrt{q})^3 R_{n-2} \leq \ldots \leq (\sqrt{q})^{n+1} R_0.$$

Now,

$$\sum_{n=1}^{\infty} \rho(x_n, x_{n+1}) = \sum_{n=1}^{\infty} R_n \leq \sum_{n=1}^{\infty} (\sqrt{q})^n R_0 = \frac{\sqrt{q} R_0}{1-\sqrt{q}} < \infty.$$

Then $\{x_n\}$ is a Cauchy sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \geq 0$. Using the α-completeness property of M we have a point $x \in M$ such that

$$\lim_{n\to\infty} x_n = x. \tag{2.8}$$

Using (2.2) and α-regularity assumption of M, we get

$$\alpha(x_n, x) \geq 1, \quad \text{for all } n. \tag{2.9}$$

If possible, suppose that for some $n \in \mathbb{N}$,

$$\frac{1}{2}\mathfrak{D}(x_n, Tx_n) > \rho(x_n, x) \quad \text{and} \quad \frac{1}{2}\mathfrak{D}(x_{n+1}, Tx_{n+1}) > \rho(x_{n+1}, x).$$

Then

$$\frac{1}{2}\rho(x_n, x_{n+1}) > \rho(x_n, x) \quad \text{and} \quad \frac{1}{2}\rho(x_{n+1}, x_{n+2}) > \rho(x_{n+1}, x).$$

Using (2.7), we have

$$\begin{aligned} R_n = \rho(x_n, x_{n+1}) &\leq \rho(x_n, x) + \rho(x, x_{n+1}) < \frac{1}{2}[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2})] \\ &= \frac{1}{2}[R_n + R_{n+1}] \leq \frac{1}{2}[R_n + R_n] = R_n, \end{aligned}$$

which leads to a contradiction. Therefore, for each $n \in \mathbb{N}$, we have

$$\text{either} \quad \frac{1}{2}\mathfrak{D}(x_n, Tx_n) \leq \rho(x_n, x) \quad \text{or} \quad \frac{1}{2}\mathfrak{D}(x_{n+1}, Tx_{n+1}) \leq \rho(x_{n+1}, x).$$

Hence, we have a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ for which

$$\frac{1}{2}\mathfrak{D}(x_{n(k)}, Tx_{n(k)}) \leq \rho(x_{n(k)}, x).$$

By (2.8), (2.9) and the above inequality, we have

$$x_{n(k)} \to x \text{ as } k \to \infty \quad \text{and} \quad \alpha(x_{n(k)}, x) \geq 1 \quad \text{for all } k.$$

Applying (iii), we get

$$\mathfrak{D}(x_{n(k)+1}, Tx) \leq \mathcal{H}(Tx_{n(k)}, Tx) \leq qQ(x_{n(k)}, x). \tag{2.10}$$

Using (2.1), we have

$$\begin{aligned} Q(x_{n(k)}, x) = \max\Big\{&\rho(x_{n(k)}, x), \mathfrak{D}(x_{n(k)}, Tx_{n(k)}), \mathfrak{D}(x, Tx), \\ &\frac{1}{2}[\mathfrak{D}(x, Tx_{n(k)}) + \mathfrak{D}(x_{n(k)}, Tx)], \sqrt{q}\frac{\mathfrak{D}(x_{n(k)}, Tx_{n(k)})\mathfrak{D}(x, Tx)}{p + \mathcal{H}(Tx_{n(k)}, Tx)}, \\ &\sqrt{q}\frac{\mathfrak{D}(x_{n(k)}, Tx)\mathfrak{D}(x, Tx_{n(k)})}{r + \mathcal{H}(Tx_{n(k)}, Tx)}\Big\} \\ \leq \max\Big\{&\rho(x_{n(k)}, x), \rho(x_{n(k)}, x_{n(k)+1}), \mathfrak{D}(x, Tx), \\ &\frac{1}{2}[\rho(x, x_{n(k)+1}) + \mathfrak{D}(x_{n(k)}, Tx)], \sqrt{q}\frac{\rho(x_{n(k)}, x_{n(k)+1})\mathfrak{D}(x, Tx)}{p + \mathfrak{D}(x_{n(k)+1}, Tx)}, \\ &\sqrt{q}\frac{\mathfrak{D}(x_{n(k)}, Tx)\rho(x, x_{n(k)+1})}{r + \mathfrak{D}(x_{n(k)+1}, Tx)}\Big\}. \end{aligned}$$

Now,

$$\begin{aligned} \limsup_{k\to\infty} Q(x_{n(k)}, x) &\leq \max\Big\{0, 0, \mathfrak{D}(x, Tx), \frac{\mathfrak{D}(x, Tx)}{2}, 0, 0\Big\} \\ &= \mathfrak{D}(x, Tx). \end{aligned} \tag{2.11}$$

Taking lim sup as $k \to \infty$ in (2.10) and applying (2.11), we have $\mathfrak{D}(x, Tx) \leq q\,\mathfrak{D}(x, Tx)$, which implies that $\mathfrak{D}(x, Tx) = 0$. Now, $\mathfrak{D}(x, Tx) = 0$ implies that $x \in \overline{Tx}$, where $\overline{Tx}$ is the closure of Tx. Since Tx is closed, we have $\overline{Tx} = Tx$. Therefore, $x \in Tx$, that is, $x \in F_T$, and so, F_T is non-empty. □

We have the following observations on Theorem 2.1.

Note 2.1. The conclusion of Theorem 2.1 is still true if one takes the assumption that T is α-continuous instead of taking the α- regularity assumption of the space. Then the portion just after (2.8) of the proof of above theorem is changed in the following way:

$$\mathfrak{D}(x, Tx) = \lim_{n\to\infty} \mathfrak{D}(x_{n+1}, Tx) \le \lim_{n\to\infty} \mathcal{H}(Tx_n, Tx) = 0.$$

Therefore, we have $\mathfrak{D}(x, Tx) = 0$, which implies that $x \in \overline{Tx} = Tx$, where $\overline{Tx}$ is the closure of Tx. Hence F_T is nonempty.

Note 2.2. The conclusion of Theorem 2.1 is still true if one considers that T is continuous instead of taking the α-regularity assumption of the spaces. Since every continuous mapping is α-continuous, the result follows from Note 2.1 and Theorem 2.1.

Note 2.3. The conclusion of Theorem 2.1 is still true if one considers that M is complete instead of taking the α-completeness assumption of M. Since every complete metric space is α-complete, the result follows from Theorem 2.1.

Example 2.1. Let $M = (-10, 10]$ and $\rho(x, y) = |x - y|$, for $x, y \in M$. Let $T\colon M \to CLB(M)$ be defined as

$$Tu = \begin{cases} \left\{\dfrac{u}{2}\right\}, & \text{if } \quad -10 < u < 0, \\ \left[0, \dfrac{u}{16}\right], & \text{if } \quad 0 \le u \le 1, \\ \{u\}, & \text{if } \quad u > 1. \end{cases}$$

Take $q = \frac{1}{4}$. Let $\alpha\colon M \times M \to [0, \infty)$ be defined as

$$\alpha(a, b) = \begin{cases} e^{a+b}, & \text{for } a \in [0, 1] \text{ and } b \in \left[0, \frac{1}{16}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\{u_n\}$ is a convergent sequence in M with limit u and $\alpha(u_n, u_{n+1}) \ge 1$, for all n. Then $u_1 \in [0, 1]$ and $u_n \in \left[0, \frac{1}{16}\right] \subseteq [0, 1]$, for $n \ge 2$. It follows that $u \in \left[0, \frac{1}{16}\right]$ and $\alpha(u_n, u) \ge 1$, for all n. Hence M has α-regular property.

Suppose $\{u_n\}$ is a Cauchy sequence in M for which $\alpha(u_n, u_{n+1}) \ge 1$, for all n. Then $u_1 \in [0, 1]$ and $u_n \in \left[0, \frac{1}{16}\right]$, for all $n \ge 2$. Then there exists $u \in \left[0, \frac{1}{16}\right]$ such that $u_n \to u$ as $n \to \infty$. Hence M is α-complete.

Take $x, y \in M$ for which $\alpha(x, y) \ge 1$. Then $0 \le x \le 1$ and $y \in \left[0, \frac{1}{16}\right]$. So, we have $Tx = \left[0, \frac{x}{16}\right] \subseteq [0, 1]$ and $Ty = \left[0, \frac{y}{16}\right] \subseteq \left[0, \frac{1}{16}\right]$. Then $\alpha(u, v) \ge 1$, whenever $u \in Tx$ and $v \in Ty$. Hence, T is α-admissible.

Here $0 \in M$, $0 \in T0$ and $\alpha(0,0) \geq 1$.

Take $u, v \in M$ for which $\alpha(u,v) \geq 1$ and $\frac{D(u,Tu)}{2} \leq \rho(u,v)$. Then $u \in [0,1]$, $v \in \left[0, \frac{1}{16}\right]$ and $\mathcal{H}(Tu,Tv) = \left|\frac{u-v}{16}\right| = \frac{\rho(u,v)}{16} = \frac{1}{4}\frac{\rho(u,v)}{4} \leq \frac{1}{4}Q(u,v)$. Therefore, all the assumptions of Theorem 2.1 are satisfied and $F_T = \{0\} \cup (1,10]$.

Note 2.4. In the above example the metric space M is α-complete but not complete. Also, the mapping T is α-continuous but not continuous.

If $\alpha(x,y) = 1$, for all $x, y \in M$, we can obtain various Suzuki-type fixed point theorems from Theorem 2.1.

Corollary 2.1. *Let (M,ρ) be a complete metric space and $0 < q < 1$. Then T has a fixed point if for $u, v \in M$, $\frac{\mathfrak{D}(u,Tu)}{2} \leq \rho(u,v)$ implies one of the following inequalities holds:*

(i) $\mathcal{H}(Tu,Tv) \leq q\,\rho(u,v)$;

(ii) $\mathcal{H}(Tu,Tv) \leq \frac{q}{2}[\mathfrak{D}(u,Tu) + \mathfrak{D}(v,Tv)]$;

(iii) $\mathcal{H}(Tu,Tv) \leq \frac{q}{2}[\mathfrak{D}(u,Tv) + \mathfrak{D}(v,Tu)]$;

(iv) $\mathcal{H}(Tu,Tv) \leq q \max\left\{\rho(u,v), \frac{\mathfrak{D}(u,Tu)+\mathfrak{D}(v,Tv)}{2}, \frac{\mathfrak{D}(u,Tv)+\mathfrak{D}(v,Tu)}{2}\right\}$.

3. Data dependence result

Let $T_1, T_2 \colon M \to CLB(M)$ be such that $\mathcal{H}(T_1x, T_2x) \leq \eta$, for all $x \in M$, where η is some positive number. A data dependence problem is to estimate the distance between the fixed point sets of these two mappings. The above is meaningful only if we have an assurance of nonempty fixed point sets of these two operators. There are also some variants of the problem. Our data dependence theorem is the following.

Theorem 3.1. *Let (M,ρ) be a metric space and $\alpha \colon M \times M \to [0,\infty)$ be a mapping such that M is α-complete and has α-regular property. Let $T_1, T_2 \colon M \to CLB(M)$ be two multivalued mappings satisfying $\mathcal{H}(T_1x, T_2x) \leq K$, for all $x \in M$, where $K > 0$ is a fixed real number. Suppose that T_2 satisfies the assumptions* (i) *and* (iii) *of Theorem 2.1. Assume that F_{T_1} is nonempty and $\alpha(x,u) \geq 1$, for all $x \in F_{T_1}$ and $u \in T_2x$. Then $F_{T_2} \neq \emptyset$ and $\sup\limits_{z \in F_{T_1}} \mathfrak{D}(z, F_{T_2}) \leq \frac{K}{q(1-\sqrt{q})}$.*

PROOF. Since $F_{T_1} \neq \emptyset$, we take $y_0 \in F_{T_1}$. Since $T_2 y_0$ is non-empty and $\alpha(y_0, u) \geq 1$, for all $u \in T_2 y_0$, T_2 satisfies the assumptions (i), (ii) and (iii) of Theorem 2.1 and hence by Theorem 2.1, F_{T_2} is non-empty, that is, $F_{T_2} \neq \emptyset$. Since $q \in (0,1)$, we have $\frac{1}{q} > 1$. As $T_1 y_0$ and $T_2 y_0 \in CLB(M)$, there exists $y_1 \in T_2 y_0$ such that

$$\rho(y_0, y_1) \leq \frac{1}{q}\mathcal{H}(T_1 y_0, T_2 y_0). \tag{3.1}$$

Now, $y_0 \in M$ and $y_1 \in T_2 y_0$ such that $\alpha(y_0, y_1) \geq 1$. Arguing similarly as in the proof of Theorem 2.1, we construct a sequence $\{y_n\}$ in M such that

$$\begin{cases} y_{n+1} \in T_2 y_n, \quad \alpha(y_n, y_{n+1}) \geq 1, \\ \rho(y_{n+1}, y_{n+2}) \leq \frac{1}{\sqrt{q}}\mathcal{H}(T_2 y_n, T_2 y_{n+1}) \\ \text{and} \quad \rho(y_{n+1}, y_{n+2}) \leq \sqrt{q}\rho(y_n, y_{n+1}) \leq \dots \\ \qquad\qquad \leq (\sqrt{q})^{n+1}\rho(y_0, y_1), \quad \text{for all } n \geq 0. \end{cases} \tag{3.2}$$

Following the arguments as in the proof of Theorem 2.1, we prove $\{y_n\}$ is a Cauchy sequence in M and there exists $\upsilon \in M$ such that

$$y_n \to \upsilon \quad \text{as } n \to \infty$$

and also υ is a fixed point of T_2, that is, $\upsilon \in T_2 \upsilon$. From (3.1), we have

$$\rho(y_0, y_1) \leq \frac{1}{q}\mathcal{H}(T_1 y_0, T_2 y_0) \leq \frac{K}{q}. \tag{3.3}$$

Using (3.2) and triangular property, we have

$$\rho(y_0, \upsilon) \leq \sum_{i=0}^{n} \rho(y_i, y_{i+1}) + \rho(y_{n+1}, \upsilon) \leq \sum_{i=0}^{n} (\sqrt{q})^i \rho(y_0, y_1) + \rho(y_{n+1}, \upsilon).$$

Letting $n \to \infty$ in the above inequality and using (3.3), we obtain

$$\rho(y_0, \upsilon) \leq \sum_{i=0}^{\infty} (\sqrt{q})^i \rho(y_0, y_1) = \frac{\rho(y_0, y_1)}{(1 - \sqrt{q})} \leq \frac{K}{q(1 - \sqrt{q})},$$

which implies that $\mathfrak{D}(y_0, F_{T_2}) \leq \frac{K}{q(1-\sqrt{q})}$. Since $y_0 \in F_{T_1}$ is arbitrary, we obtain $\sup_{z \in F_{T_1}} \mathfrak{D}(z, F_{T_2}) \leq \frac{K}{q(1-\sqrt{q})}$. □

4. Stability analysis

Stability is related limiting behavior of a system which, in this case, is the relation of the fixed point sets associated with a sequence of multivalued mappings with the limit function to which the sequence converges. Let $\{T_n\colon M \to CLB(M)\}$ be a sequence of multivalued mappings that converges to a mapping $T\colon M \to CLB(M)$, that is, $T = \lim_{n\to\infty} T_n$. Suppose that $\{F_{T_n}\}$ is the sequence of fixed point sets of the sequence of mappings $\{T_n\}$ and F_T is the fixed point set of T. We say that the fixed point sets F_{T_n} of the sequence of multivalued mappings $\{T_n\colon M \to CLB(M)\}$ are stable if $\mathcal{H}(F_{T_n}, F_T) \to 0$ as $n \to \infty$.

In continuation of the data dependence result of the previous section, by particularly considering a special case in which both the mappings are assumed to satisfy the conditions of the main theorem in Section 2, we establish a stability result for fixed point sets of these mappings.

Lemma 4.1. *Let (M,ρ) be a metric space and $\alpha\colon M \times M \to [0,\infty)$. Let $\{T_n\colon M \to CLB(M) : n \in \mathbb{N}\}$ be a sequence of multivalued mappings converging to a mapping $T\colon M \to CLB(M)$. If each T_n $(n \in \mathbb{N})$ is a Suzuki-type α-contraction, then T is also a Suzuki-type α-contraction.*

Proof. Take $x, y \in M$ for which $\alpha(x,y) \geq 1$. Since each T_n $(n \in \mathbb{N})$ is a Suzuki-type α- contraction, we have

$$\frac{1}{2}\mathfrak{D}(x, T_n x) \leq \rho(x,y) \quad \text{implies}$$

$$\mathcal{H}(T_n x, T_n y) \leq q \max\Big\{\rho(x,y), \mathfrak{D}(x,T_n x), \mathfrak{D}(y, T_n y), \frac{1}{2}[\mathfrak{D}(x,T_n y) + \mathfrak{D}(y, T_n x)],$$

$$\sqrt{q}\frac{\mathfrak{D}(x,T_n x)\mathfrak{D}(y,T_n y)}{p + \mathcal{H}(T_n x, T_n y)}, \sqrt{q}\frac{\mathfrak{D}(x,T_n y)\mathfrak{D}(y,T_n x)}{r + \mathcal{H}(T_n x, T_n y)}\Big\}.$$

Taking limit as $n \to \infty$ in the above inequalities, we get

$$\frac{1}{2}\mathfrak{D}(x, T x) \leq \rho(x,y) \quad \text{implies}$$

$$\mathcal{H}(T x, T y) \leq q \max\Big\{\rho(x,y), \mathfrak{D}(x,T x), \mathfrak{D}(y, T y), \frac{1}{2}[\mathfrak{D}(x,T y) + \mathfrak{D}(y, T x)],$$

$$\sqrt{q}\frac{\mathfrak{D}(x,T x)\mathfrak{D}(y,T y)}{p + \mathcal{H}(T x, T y)}, \sqrt{q}\frac{\mathfrak{D}(x,T y)\mathfrak{D}(y,T x)}{r + \mathcal{H}(T x, T y)}\Big\}.$$

This shows that T is a Suzuki-type α-contraction. □

Theorem 4.1. *Let (M,ρ) be a metric space and $\alpha\colon M\times M\to[0,\infty)$ be a mapping such that M is α-complete and has α-regular property. Let $\{T_n\colon M\to CLB(M) : n\in\mathbb{N}\}$ be a sequence of multivalued mappings converging to $T\colon M\to CLB(M)$ uniformly, that is, $T_n\to T$ uniformly as $n\to\infty$. Suppose that each T_n $(n\in\mathbb{N})$ satisfies the assumptions* (i), (ii) *and* (iii) *of Theorem* 2.1 *and also T satisfies the assumptions* (i) *and* (ii) *of Theorem* 2.1. *Then $F_{T_n}\neq\emptyset$ for every n and $F_T\neq\emptyset$. If $\alpha(x,u)\geq 1$, whenever $x\in F_{T_n}$ $(n\in\mathbb{N})$ and $u\in Tx$ or $x\in F_T$ and $u\in T_nx$ $(n\in\mathbb{N})$, then the fixed point sets of the sequence $\{T_n\}$ are stable.*

Proof. By Theorem 2.1, $F_{T_n}\neq\emptyset$ for every $n\in\mathbb{N}$. By Lemma 4.1 and Theorem 2.1, $F_T\neq\emptyset$. Let $K_n=\sup_{x\in X}\mathcal{H}(T_nx,Tx)$, where $n\in\mathbb{N}$. Since the sequence $\{T_n\}$ is uniformly convergent to T, we have

$$\lim_{n\to\infty}K_n=\lim_{n\to\infty}\sup_{x\in X}\mathcal{H}(T_nx,Tx)=0. \tag{4.1}$$

By Theorem 3.1, we obtain

$$\sup_{z\in F_T}\mathfrak{D}(z,F_{T_n})\leq\frac{K_n}{q(1-\sqrt{q})}\quad\text{and}\quad\sup_{z\in F_{T_n}}\mathfrak{D}(z,F_T)\leq\frac{K_n}{q(1-\sqrt{q})}.$$

Therefore, we have

$$\mathcal{H}(F_{T_n},F_T)\leq\frac{K_n}{q(1-\sqrt{q})},\quad\text{for all } n\in\mathbb{N}.$$

Taking limit as $n\to\infty$ in the above inequality and using (4.1), we get $\lim_{n\to\infty}\mathcal{H}(F_{T_n},F_T)=0$. Therefore, the fixed point sets of mappings of the sequence $\{T_n\}$ are stable. □

5. Some results on graphic contraction

In the present section, we extend our results in metric spaces with an additional structure of graph. Suppose that the metric space (M,ρ) is endowed with a directed graph $\mathfrak{G}(V,E)$, that is, $\mathfrak{G}$ is a directed graph such that its vertex set $V(\mathfrak{G})$ coincides with M and the edge set $E(\mathfrak{G})$ contains all loops. Assume that $\mathfrak{G}$ has no parallel edges.

Fixed point problem on the structures of metric spaces with a graph is a recent development. Works like [1, 15, 20] are some instances.

Definition 5.1. A multivalued mapping $T\colon M \to CLB(M)$ is called $\mathfrak{G}$-*admissible* if $(x, y) \in E$ for $x, y \in M$ implies $(u, v) \in E$, whenever $u \in Tx$, $v \in Ty$.

Definition 5.2. M is called $\mathfrak{G}$-*regular* if $(x_n, x) \in E$ for all n, whenever $\{x_n\}$ is a convergent sequence in M with limit x and $(x_n, x_{n+1}) \in E$ for all n.

Definition 5.3. A multivalued mapping $T\colon M \to CLB(M)$ is called $\mathfrak{G}$-*continuous at* $x \in M$ if $\lim_{n\to\infty} \mathcal{H}(Tx_n, Tx) = 0$, whenever $\{x_n\}$ is any convergent sequence in M having limit x and $(x_n, x_{n+1}) \in E$ for all n.

Definition 5.4. M is called $\mathfrak{G}$-*complete* if every Cauchy sequence $\{x_n\}$ in M with $(x_n, x_{n+1}) \in E$ for all n converges in M.

Definition 5.5. A multivalued mapping $T\colon M \to CLB(M)$ is said to be a *Suzuki-type graphic contraction* if for all $u, v \in M$ with $(u, v) \in E$,

$$\frac{1}{2}\mathfrak{D}(u, Tu) \le \rho(u, v) \quad \text{implies} \quad \mathcal{H}(Tu, Tv) \le q\, Q(u, v),$$

where $Q(u, v)$, q, p and r are as in Definition 1.7.

Theorem 5.1. *Let (M, ρ) be a metric space endowed with a directed graph $\mathfrak{G}(V, E)$ such that M is $\mathfrak{G}$-complete and has $\mathfrak{G}$-regular property. Let $T\colon M \to CLB(M)$ be such that* (i) *T is $\mathfrak{G}$-admissible,* (ii) *there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$,* (iii) *T is a Suzuki-type $\mathfrak{G}$-contraction. Then F_T is non-empty.*

Proof. Define $\alpha\colon M \times M \to [0, \infty)$ as $\alpha(u, v) = \begin{cases} 1, \text{ if } (u, v) \in E, \\ 0, \text{ if } (u, v) \notin E. \end{cases}$

It can be easily verified that all the assumptions of Theorem 2.1 are satisfied and hence F_T is non-empty. □

Conclusion. In the expanding scenario of research on fixed point theory it is worth seeing how different lines of study coalesce amongst themselves to create new results. In the present paper we have made such an attempt. We think that more of such efforts can enrich the theory of fixed points in a substantial way.

The constant q which is taken in the Suzuki-type α-contraction considered in Theorem 2.1 may be replaced by a Mizoguchi-Takahashi function ([26]). Here we have not proceeded with it but this can be taken up as an immediate future work. The investigation of possible application of the corresponding theorem to integral and differential inclusion problem is supposed to be of considerable interest. The study of different types of stability such as Ulam–Hyers stability, asymptotic stability, etc., error estimation and rate of convergence of fixed point sets in the current context or in similar contexts would be an interesting topic for future study.

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