

## GREEN'S FUNCTIONS FOR INTERIOR AND EXTERIOR HELMHOLTZ PROBLEMS

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**Abstract.** In the paper, the derivation of Green's functions for Helmholtz equation in circular, annular and exterior circular domains is presented. The Green's functions are assumed of the form a cosine series. An example of application of the Green's functions in frequency analysis of a composite membrane is presented.

### Introduction

The Green's functions were successfully applied to solving various differential problems formulated in mathematical modeling of physical processes [1-3]. Particularly, the class of problems describing with Helmholtz equation can be solved by using the Green's function method. The first stage in application of the method is deriving the necessary Green's functions.

Derivations of the Green's functions for various differential equations are presented in many monographs and articles [1-3]. The Green's functions for the two-dimensional the Helmholtz equation over rectangular domain with Dirichlet and Neumann boundary conditions are given in the book [1]. Also the Green's functions for Helmholtz equation when the domain consists of the annular rings with Dirichlet conditions or Neumann conditions along the inner and outer boundaries are presented in this book. Unfortunately, the Green's functions over annular rings for other combination of Dirichlet and Neumann conditions at inner and outer boundaries are not given. These Green's functions are needed for deriving a solution to the vibration problem of a composite circular and annular membrane consisting with annular rings, each of constant density [4, 5]. In the paper [3] the Green's function for the Helmholtz differential equation with an impedance boundary condition is discussed.

In this paper the derivation of the Green's functions for interior Helmholtz equations and free space Green's functions for exterior Helmholtz equations is presented.

The Green's functions for the Helmholtz equation defined over the plane (free space Green's functions) are applied to finding an approximate solution of Helmholtz problem by using the fundamental solution method [6, 7]. We consider here

the domain  $D = \mathbb{R}^2 \setminus K$ , where  $K$  is a circle. The free space Green's functions in two cases are presented: the Green's functions which satisfies Dirichlet or Neumann conditions at the boundary  $\partial K$ .

## 1. Green's functions for interior Helmholtz problems

Let us consider the Helmholtz equation in a circular or an annular domain  $D$

$$\frac{\partial^2 G_H}{\partial r^2} + \frac{1}{r} \frac{\partial G_H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G_H}{\partial \theta^2} + \lambda^2 G_H = \frac{\delta(r-\zeta)\delta(\theta-\theta')}{r}, \quad (r, \theta) \in D \quad (1)$$

The solution of this equation can be obtained by using the method of separation of variables. In our consideration we assume function  $G_H$  in the form:

$$G_H(r, \theta, \zeta, \theta') = \sum_{n=-\infty}^{\infty} G_n(r, \zeta) \cos n(\theta - \theta') \quad (2)$$

This leads to differential equation and boundary conditions respect to variable  $r$ . Green's function  $G_n(r, \zeta)$  for this boundary problem satisfies a non-homogeneous equation

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dG_n(r, \zeta)}{dr} \right) + \left( \lambda^2 - \frac{n^2}{r^2} \right) G_n(r, \zeta) = \frac{-\delta(r-\zeta)}{2\pi r} \quad (3)$$

and the homogeneous boundary conditions. We consider equation (3) in circular or annular domains. We assume Dirichlet or Neumann boundary conditions.

The solution of this equation can be presented in the form [7]

$$G_n(r, \zeta) = g_{0n}(r, \zeta) + g_{1n}(r, \zeta) H(r - \zeta) \quad (4)$$

The function  $g_{0n}$  is the general solution of the homogeneous equation, which is obtained from equation (3) by replacing of the right-hand side with the zero function. The solution has the form

$$g_{0n}(r, \zeta) = c_1 J_n(\lambda r) + c_2 Y_n(\lambda r) \quad (5)$$

The constants of integration  $c_1$  and  $c_2$  are determined by taking into account (4) in boundary conditions.

The function  $g_{1n}$  occurring in equation (4), is a solution of the same homogeneous differential equation and satisfies the following conditions [2]

$$g_{1n}(r, \zeta) \Big|_{r=\zeta} = 0, \quad \frac{\partial}{\partial r} g_{1n}(r, \zeta) \Big|_{r=\zeta} = \frac{1}{\zeta} \quad (6)$$

Using the function (5) in conditions (6), after transformation, one obtains

$$g_{10}(r, \zeta) = \frac{\pi}{2} \left[ J_n(\lambda \zeta) Y_n(\lambda r) - J_n(\lambda r) Y_n(\lambda \zeta) \right] \quad (7)$$

### **Circular region with Neumann boundary condition**

For the circular region the Neumann boundary condition is

$$\left. \frac{\partial G_n(r, \zeta)}{\partial r} \right|_{r=b} = 0 \quad (8)$$

In this case we have  $c_2 = 0$  in equation (5) because we assume that  $|G_n(0, \zeta)| < \infty$ . Taken into account the boundary conditions (8), we obtain the function  $g_{0n}$  in the form

$$g_{0n}(r, \zeta) = \frac{\pi}{2} \left[ Y_n(\lambda \zeta) - \frac{p_n(b)}{q_n(b)} J_n(\lambda \zeta) \right] J_n(\lambda r) \quad (9)$$

where:  $p_n(r) = Y_{n+1}(\lambda r) - Y_{n-1}(\lambda r)$ ,  $q_n(r) = J_{n+1}(\lambda r) - J_{n-1}(\lambda r)$ .

Finally, using equations (4), (7) and (9), the Green's function  $G_n(r, \zeta)$  for circular region with the Neumann boundary condition at  $r = b$ , it can be written as (Fig. 1a)

$$G_n(r, \zeta) = \begin{cases} \frac{\pi}{2} \left[ Y_n(\lambda \zeta) - \frac{p_n(b)}{q_n(b)} J_n(\lambda \zeta) \right] J_n(\lambda r) & \text{for } 0 \leq r < \zeta \leq b \\ \frac{\pi}{2} \left[ Y_n(\lambda r) - \frac{p_n(b)}{q_n(b)} J_n(\lambda r) \right] J_n(\lambda \zeta) & \text{for } 0 \leq \zeta \leq r \leq b \end{cases} \quad (10)$$

### **Annular region with Dirichlet-Neumann boundary conditions**

The Dirichlet and Neumann conditions at boundary of an annular region:  $r = a$  and  $r = b$ , respectively, are

$$G_n(r, \zeta)|_{r=a} = 0, \quad \left. \frac{\partial}{\partial r} G_n(r, \zeta) \right|_{r=b} = 0, \quad a \leq r \leq b \quad (11)$$

The Green's function is defined by equation (4) with the function  $g_{1n}$  given by equation (7). The function  $g_{0n}$  is determined by using the boundary conditions (11). This function is as follows

$$g_{0n}(r, \zeta) = \frac{\pi [Y_n(\lambda \zeta) q_n(b) - J_n(\lambda \zeta) p_n(b)] [Y_n(\lambda a) J_n(\lambda r) - J_n(\lambda a) Y_n(\lambda r)]}{2 [Y_n(\lambda a) q_n(b) - J_n(\lambda a) p_n(b)]} \quad (12)$$

The Green's function  $G_n(r, \zeta)$  for the annular region with Dirichlet-Neumann boundary conditions using equations (4), (7) and (12) we obtain in the form (Fig. 1b)

$$G_n(r, \zeta) = \begin{cases} \frac{\pi \phi_n(\zeta, b)}{2 \phi_n(a, b)} [Y_n(\lambda a) J_n(\lambda r) - J_n(\lambda a) Y_n(\lambda r)] & \text{for } a \leq r < \zeta \leq b \\ \frac{\pi \phi_n(r, b)}{2 \phi_n(a, b)} [Y_n(\lambda a) J_n(\lambda \zeta) - J_n(\lambda a) Y_n(\lambda \zeta)] & \text{for } a \leq \zeta \leq r \leq b \end{cases} \quad (13)$$

where  $\phi_n(\alpha, \beta) = Y_n(\lambda \alpha) q_n(\beta) - J_n(\lambda \alpha) p_n(\beta)$ .

#### *Annular region with Neumann-Neumann boundary conditions*

The Green's function  $G_n(r, \zeta)$  for the Helmholtz equation with Neumann-Neumann conditions is given by equations (4) and (7). The boundary conditions are:

$$\left. \frac{\partial}{\partial r} G_n(r, \zeta) \right|_{r=a} = \left. \frac{\partial}{\partial r} G_n(r, \zeta) \right|_{r=b} = 0, \quad a \leq r \leq b \quad (14)$$

The function  $g_{0n}$  obtained by using the boundary conditions (14) has the form:

$$g_{0n}(r, \zeta) = \frac{\pi [Y_n(\lambda \zeta) q_n(b) - J_n(\lambda \zeta) p_n(b)] [Y_n(\lambda r) q_n(a) - J_n(\lambda r) p_n(a)]}{2 [p_n(b) q_n(a) - p_n(a) q_n(b)]} \quad (15)$$

and the Green's function corresponding to these conditions is (Fig. 1c)

$$G_n(r, \zeta) = \begin{cases} \frac{\pi \phi_n(\zeta, b) \phi_n(r, a)}{2 [p_n(b) q_n(a) - p_n(a) q_n(b)]} & \text{for } a \leq r < \zeta \leq b \\ \frac{\pi \phi_n(r, b) \phi_n(\zeta, a)}{2 [p_n(b) q_n(a) - p_n(a) q_n(b)]} & \text{for } a \leq \zeta \leq r \leq b \end{cases} \quad (16)$$

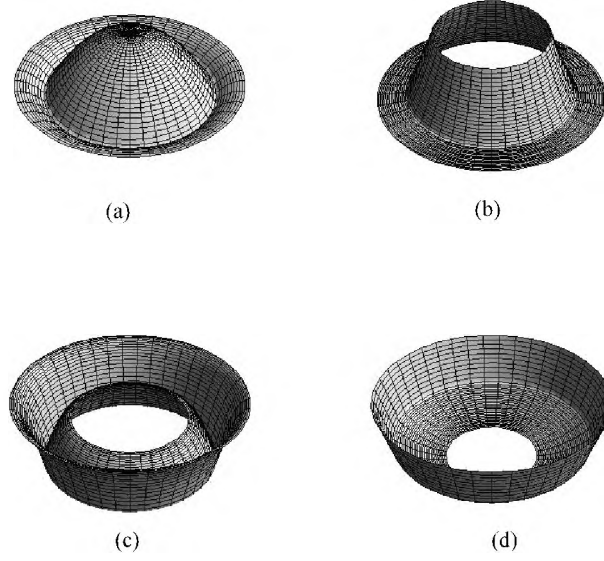


Fig. 1. The Green's functions for: a) the circular region with Neumann boundary condition, b) the annular region with Dirichlet-Neumann boundary conditions, c) the annular region with Neumann-Neumann boundary conditions, d) the annular region with Neumann-Dirichlet boundary conditions,  $n = 0$ ,  $\Omega = 2$

#### ***Annular region with Neumann-Dirichlet boundary conditions***

For boundary conditions:

$$\left. \frac{\partial}{\partial r} G_n(r, \zeta) \right|_{r=a} = 0, \quad G_n(r, \zeta) \Big|_{r=b} = 0, \quad a \leq r \leq b \quad (17)$$

the function  $g_{0n}$  has the form:

$$g_{0n}(r, \zeta) = \frac{\pi [J_n(\lambda \zeta) Y_n(\lambda b) - J_n(\lambda b) Y_n(\lambda \zeta)] [Y_n(\lambda r) q_n(a) - J_n(\lambda r) p_n(a)]}{-2 [Y_n(\lambda b) q_n(a) - J_n(\lambda b) p_n(a)]} \quad (18)$$

and the Green's function is (Fig. 1d)

$$G_n(r, \zeta) = \begin{cases} -\frac{\pi \phi_n(r, a)}{2 \phi_n(b, a)} [J_n(\lambda \zeta) Y_n(\lambda b) - J_n(\lambda b) Y_n(\lambda \zeta)] & \text{for } a \leq r < \zeta \leq b \\ -\frac{\pi \phi_n(\zeta, a)}{2 \phi_n(b, a)} [J_n(\lambda r) Y_n(\lambda b) - J_n(\lambda b) Y_n(\lambda r)] & \text{for } a \leq \zeta \leq r \leq b \end{cases} \quad (19)$$

## 2. Green's functions for exterior Helmholtz problems

The solution of the Helmholtz problem in a bounded domain can be obtained by using the free space Green's function in the method of fundamental solutions [4, 5]. The function for problems in 2D domain has the form:

$$G_n(P, Q) = \frac{i}{4} H_n^{(1)}(\lambda d(P, Q)) \quad (20)$$

where  $H_0^{(1)}(\cdot)$  is the Hankel function of the first kind and zero order.  $d(P, Q) = \sqrt{(x - \xi)^2 + (y - \eta)^2}$  where  $P = (x, y)$ ,  $Q = (\xi, \eta)$ ,  $i = \sqrt{-1}$ . In polar coordinates we have:  $x = r \cos \varphi$ ,  $y = r \sin \varphi$ ,  $\xi = \rho \cos \vartheta$ ,  $\eta = \rho \sin \vartheta$  and  $d(P, Q) = \sqrt{r^2 + \rho^2 - 2r\rho \cos(\varphi - \vartheta)}$ .

We present below the free space Green's function for Helmholtz equation considered in the infinite region  $D$  exterior to a centered circle with radius  $a$ . We assume at the boundary  $\partial D$  the Dirichlet or the Neumann condition. These functions can be derived by the use of separation of variables as was shown in the chapter 1 for interior Green's functions. In this case the one-dimensional free space Green's function can be determined as a solution of equation (3) for  $r > a$ .

The general solution of the problem has the form given by equation (5). The function  $g_{ln}(r, \zeta)$  is also the same as in interior problems and is written by equation (7).

The Dirichlet boundary condition at  $r = a$  leads to the relationship

$$c_2 = -\frac{J_n(\lambda a)}{Y_n(\lambda a)} c_1 \quad (21)$$

Therefore, the free space Green's function for exterior Helmholtz problems has the form (Fig. 2a):

$$G_n(r, \zeta) = c_1 \left[ J_n(\lambda r) - \frac{J_n(\lambda a) Y_n(\lambda r)}{Y_n(\lambda a)} \right] + g_{ln}(r, \zeta) H(r - \frac{r}{a}) \quad (22)$$

The constant  $c_1$  can be determined by using a condition at infinity. Using the free space Green's function in the method of fundamental solutions the constant can be assumed as an arbitrary value.

Similarly, for Neumann condition at  $r = a$  we obtain

$$c_2 = -\frac{J_{n-1}(\lambda a) - J_{n+1}(\lambda a)}{Y_{n-1}(\lambda a) - Y_{n+1}(\lambda a)} c_1 \quad (23)$$

and the free space Green's function can be presented in the form (Fig. 2b):

$$G_n(r, \zeta) = c_1 \left[ J_n(\lambda r) - \frac{J_{n-1}(\lambda a) - J_{n+1}(\lambda a)}{Y_{n-1}(\lambda a) - Y_{n+1}(\lambda a)} Y_n(\lambda r) \right] + g_{1n}(r, \zeta) H(r - \zeta) \quad (24)$$



Fig. 2. The Green's functions for exterior Helmholtz problems (a) with the Dirichlet boundary condition, (b) with the Neumann boundary condition,  $n = 0$ ,  $\Omega = 2$

### 3. Example of application

Let us consider the vibration of a circular membrane consisting with circular and annular regions presented in Figure 3. The transverse vibration of the membrane segments are governed by differential equations

$$s \left( \frac{\partial^2 u_k}{\partial r^2} + \frac{1}{r} \frac{\partial u_k}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_k}{\partial \theta^2} \right) - \rho_k \frac{\partial^2 u_k}{\partial t^2} = 0, \quad k = 1, 2 \quad (25)$$

where  $s$  is the tension per unit length,  $\rho_k$  is the density of  $k$ -th segment of the membrane,  $r, \theta$  are polar coordinates and  $t$  is time.

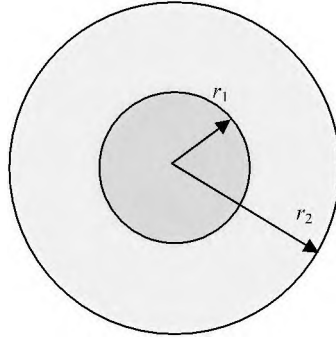


Fig. 3. A sketch of a circular membrane consisting with circular and annular regions

The functions  $u_k$  satisfy the continuity conditions at  $r = r_1$

$$u_1(r_1, \theta, t) = u_2(r_1, \theta, t) \quad (26)$$

$$\left. \frac{\partial u_1(r, \theta, t)}{\partial r} \right|_{r=r_1} = \left. \frac{\partial u_2(r, \theta, t)}{\partial r} \right|_{r=r_1} \quad (27)$$

and boundary condition:

$$u_2(r_2, \theta, t) = 0 \quad (28)$$

Assuming the functions  $u_k$  in (25)-(28) in the form

$$u_k(r, \theta, t) = U_{kn}(r) \cos n\theta \cos \omega t \quad (29)$$

we obtain the following boundary-value problem:

$$\begin{cases} \frac{1}{r} \frac{d}{dr} \left( r \frac{dU_{1n}(r)}{dr} \right) + \left( \lambda_1^2 - \frac{n^2}{r^2} \right) U_{1n}(r) = 0 \\ \frac{1}{r} \frac{d}{dr} \left( r \frac{dU_{2n}(r)}{dr} \right) + \left( \lambda_2^2 - \frac{n^2}{r^2} \right) U_{2n}(r) = 0 \end{cases} \quad (30)$$

$$U_{1n}(r_1) - U_{2n}(r_1), \quad \left. \frac{dU_{1n}(r)}{dr} \right|_{r=r_1} - \left. \frac{dU_{2n}(r)}{dr} \right|_{r=r_1} \quad (31)$$

$$U_{2n}(r_2) = 0 \quad (32)$$

where  $\lambda_1 = \omega \sqrt{\frac{\rho_1}{s}}$ ,  $\lambda_2 = \omega \sqrt{\frac{\rho_2}{s}}$ .

The solution of the problem (30)-(32) as is shown in paper [6] may be presented with the use of the Green's functions for the Helmholtz problem in the circular region with Neumann boundary condition (4), (7), (9) and in the annular region with Neumann-Dirichlet boundary conditions (4), (7), (16)

$$\begin{aligned} U_{1n}(r) &= S_{1n} G_{1n}(r, r_1), & 0 \leq r \leq r_1 \\ U_{2n}(r) &= -S_{1n} G_{2n}(r, r_1), & r_1 < r \leq r_2 \end{aligned} \quad (33)$$

Functions  $U_{1n}, U_{2n}$  satisfy the conditions (31b), (32). Taking into account equations (33) in condition (31a) we obtain



$$S_{in} [G_{1n}(r_1, r_1) + G_{2n}(r_1, r_1)] = 0 \quad (34)$$

This equation is fulfilled when

$$G_{1n}(r_1, r_1) + G_{2n}(r_1, r_1) = 0 \quad (35)$$

Equation (35) is the frequency equation which is solved numerically with respect to  $\omega_n$ .

Numerical results of the non-dimensional free vibration frequencies  $\Omega_n = \omega_n r_2 \sqrt{\frac{\rho_2}{s}}$  for the various values of  $\sigma = \frac{\rho_1}{\rho_2}$  are presented in Table 1.

Table 1

**Frequency parameter values  $\Omega_{nk}$  of the composite circular membrane shown in Figure 3 for various values of  $\sigma = \rho_1/\rho_2$  ( $r_1/r_2 = 0.5$ )**

$\sigma$	$\Omega_{01}$	$\Omega_{02}$	$\Omega_{03}$	$\Omega_{04}$	$\Omega_{05}$
0.1	3.57325	9.55645	15.73338	21.94047	28.14370
1.0	2.40483	5.52008	8.65373	11.79153	14.93092
5.0	0.57109	1.64999	2.84236	4.04593	5.20686
10.0	0.28701	0.83304	1.43825	2.05526	2.67589

## Conclusions

The Green's functions which are used in solving the Helmholtz equation occurring in vibration problems of composite membranes have been derived. The closed form of the Green's function is obtained as a solution of the boundary problem for circular and annular domains for various boundary conditions. For the exterior Helmholtz problem the free space Green's functions have been presented. These functions can be used in the method of fundamental solutions applying to Helmholtz equation in domains with a circular hole. The presented example shows the use of the Green's functions in vibration analysis of a composite membrane consisting in circular and annular regions.

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