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THE NUMERICAL SOLUTION OF THE TRANSIENT HEAT CONDUCTION PROBLEM USING THE LATTICE BOLTZMANN METHOD

Grażyna Kałuża

Silesian University of Technology, Poland
grazyna.kaluza@polsl.pl

Abstract. The implementation of the lattice Boltzmann method (LBM) for the solution of the transient heat conduction problem is presented. The one dimensional task is considered and the different boundary conditions, specifically the Dirichlet, Neumann and Robin ones are taken into account. The D1Q2 lattice model is applied. To check the accuracy of the LBM algorithm, the same problems have been solved using the explicit variant of the finite difference method. In the final part of the paper, the results of computations are shown and the conclusions are formulated.

Introduction

Over the last decade the lattice Boltzmann method (LBM) has been developed as a promising computational tool to analyze the large class of engineering problems, among others, the heat transfer problems [1, 2]. In this paper the LBM has been applied in order to solve the Fourier equation

$$X \in D: \quad c\frac{\partial T(X,t)}{\partial t} = \lambda \nabla^2 T(X,t) + Q(X,t) \tag{1}$$

where λ is the thermal conductivity, c is the volumetric specific heat, $Q(X, t)$ is the source function, T, X, t denote the temperature, spatial co-ordinates and time. The equation (1) is supplemented by boundary conditions

$$\begin{aligned} &X \in \Gamma_1: \quad T(X,t) = T_b \\ &X \in \Gamma_2: \quad q(X,t) = -\lambda\frac{\partial T(X,t)}{\partial n} = q_b \\ &X \in \Gamma_3: \quad q(X,t) = -\lambda\frac{\partial T(X,t)}{\partial n} = \alpha\left[T(X,t) - T_a\right] \end{aligned} \tag{2}$$

and the initial one

$$t \quad 0 : \quad T(X,0) \quad T_p \tag{3}$$

where $\partial T/\partial n$ denotes the normal derivative, n is the normal outward vector. T_b, q_b are the known boundary temperature and boundary heat flux, respectively, α is the heat transfer coefficient, T_a is the ambient temperature and T_p is the initial temperature.

1. Kinetic equation

The Boltzmann transport equation can be written as [2]

$$\frac{\partial f}{\partial t} + \mathbf{e} \cdot \nabla f = \Omega \tag{4}$$

where f is the distribution function, $\mathbf{e}$ is the velocity and Ω is the collision operator. It should be pointed out that it is difficult to solve equation (4) because Ω is a function of f and in a general case it is an integro-differential equation [2].

The starting point of the lattice Boltzmann method (LBM) is the kinetic equation [1, 3, 4]

$$\frac{\partial f_i(X,t)}{\partial t} + e_i \cdot \nabla f_i(X,t) = \Omega_i, \quad i = 1,2,3,\ldots M \tag{5}$$

where f_i is the particle distribution function denoting the number of particles at the lattice node x at the time t moving in direction i with the velocity e_i along the lattice link $h = e_i \Delta t$ connecting the nearest neighbors and M is the number of directions in a lattice through which the information propagates. The term Ω_i represents the rate of change of f_i due to collisions and is very complicated [2].

The simplest model for Ω_i is the Bhatnagar-Gross-Krook (BGK) approximation [1, 4]

$$\Omega_i = -\frac{1}{\tau}\left[f_i(X,t) - f_i^0(X,t)\right] \tag{6}$$

where τ is the relaxation time and $f_i^0(X, t)$ is the equilibrium distribution function.

It should be pointed out that in the case of heat transfer problems the equilibrium distribution function is given by

$$f_i^0(X,t) = w_i T(X,t) \tag{7}$$

where w_i are the known weights, at the same time $\sum_{i=1}^{M} w_i = 1$.

Additionally, the temperature T at the lattice node x and for time t is calculated using the formula [1, 3]

$$T(X,t)=\sum_{i=1}^{M} f_i(X,t) \tag{8}$$

From equations (7), (8) results that

$$\sum_{i=1}^{M} f_i^0(X,t)=\sum_{i=1}^{M} w_i T(X,t)=T(X,t)=\sum_{i=1}^{M} f_i(X,t) \tag{9}$$

2. Lattice Boltzmann method for 1D problem

In this paper, for 1D problem, D1Q2 lattice model [1, 5] has been used as shown in Figure 1. Nodes 0 and n are the boundary ones, while the nodes 1, 2, … $n-1$ are the internal ones.

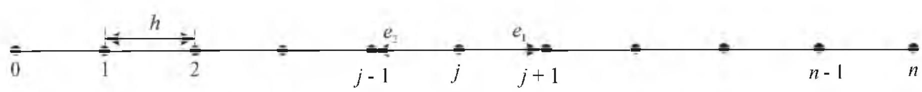

Fig. 1. D1Q2 lattice

In such case the lattice Boltzmann transport equation can be written as [4]

$$\frac{\partial f_i(x,t)}{\partial t}+e_i\frac{\partial f_i(x,t)}{\partial x}=-\frac{1}{\tau}\left[f_i(x,t)-f_i^0(x,t)\right]+w_i\frac{Q(x,t)}{c},\quad i=1,\ 2 \tag{10}$$

For the D1Q2 lattice, the two velocities e_i and their corresponding weights w_i are as follows

$$e_1=v,\quad e_2=-v$$
$$w_1=w_2=\frac{1}{2} \tag{11}$$

at the same time $v=h/\Delta t$ (h is the lattice distance from node to node).
The relaxation time τ is computed from [1]

$$\tau=\frac{a}{v^2}+\frac{\Delta t}{2} \tag{12}$$

where $a=\lambda/c$ is the thermal diffusivity and Δt is the time step.

In other words, two equations should be solved, namely

$$\frac{\partial f_1(x,t)}{\partial t}+v\frac{\partial f_1(x,t)}{\partial x}=-\frac{1}{\tau}\left[f_1(x,t)-f_1^0(x,t)\right]+w_1\frac{Q(x,t)}{c} \tag{13}$$

$$\frac{\partial f_2(x,t)}{\partial t}-v\frac{\partial f_2(x,t)}{\partial x}=-\frac{1}{\tau}\left[f_2(x,t)-f_2^0(x,t)\right]+w_2\frac{Q(x,t)}{c} \tag{14}$$

The approximation of the first derivatives using right-hand side differential quotients is the following [2, 6, 7]

$$\begin{gathered}\frac{\partial f_1}{\partial t}=\frac{f_1(x,t+\Delta t)-f_1(x,t)}{\Delta t}\\ \frac{\partial f_1}{\partial x}=\frac{f_1(x+h,t-\Delta t)-f_1(x,t+\Delta t)}{h}\end{gathered} \tag{15}$$

and using left-hand side differential quotients is of the form

$$\begin{gathered}\frac{\partial f_2}{\partial t}=\frac{f_2(x,t+\Delta t)-f_2(x,t)}{\Delta t}\\ \frac{\partial f_2}{\partial x}=\frac{f_2(x,t+\Delta t)-f_2(x-h,t+\Delta t)}{h}\end{gathered} \tag{16}$$

Introducing (15) into (13) and (16) into (14), respectively, one obtains

$$\begin{gathered}\frac{f_1(x,t-\Delta t)-f_1(x,t)}{\Delta t}+v\frac{f_1(x+h,t+\Delta t)-f_1(x,t+\Delta t)}{h}=\\ =-\frac{1}{\tau}\left[f_1(x,t)-f_1^0(x,t)\right]+w_1\frac{Q(x,t)}{c}\end{gathered} \tag{17}$$

and

$$\begin{gathered}\frac{f_2(x,t+\Delta t)-f_2(x,t)}{\Delta t}-v\frac{f_2(x,t+\Delta t)-f_2(x-h,t+\Delta t)}{h}=\\ =-\frac{1}{\tau}\left[f_2(x,t)-f_2^0(x,t)\right]+w_2\frac{Q(x,t)}{c}\end{gathered} \tag{18}$$

From equations (17) and (18) results that

$$\begin{gathered}f_1(x+h,t+\Delta t)=\left(1-\frac{\Delta t}{\tau}\right)f_1(x,t)+\frac{\Delta t}{\tau}f_1^0(x,t)+w_1\,\Delta t\,\frac{Q(x,t)}{c}\\ f_2(x-h,t+\Delta t)=\left(1-\frac{\Delta t}{\tau}\right)f_2(x,t)+\frac{\Delta t}{\tau}f_2^0(x,t)+w_2\,\Delta t\,\frac{Q(x,t)}{c}\end{gathered} \tag{19}$$

this means

$$
\begin{aligned}
f_{1\,j+1}^{p+1} &= \left(1-\frac{\Delta t}{\tau}\right) f_{1\,j}^{p} + \frac{\Delta t}{\tau} f_{1\,j}^{0\,p} + w_1\,\Delta t\,\frac{Q_j^p}{c}, \quad j = 0, 1, \dots, n-1 \\
f_{2\,j-1}^{p+1} &= \left(1-\frac{\Delta t}{\tau}\right) f_{2\,j}^{p} + \frac{\Delta t}{\tau} f_{2\,j}^{0\,p} + w_2\,\Delta t\,\frac{Q_j^p}{c}, \quad j = n, n-1, \dots, 1
\end{aligned}
\tag{20}
$$

In equations (20) (c.f. formula (7)):

$$
f_{1\,j}^{0\,p} = w_1\,T_j^p, \quad f_{2\,j}^{0\,p} = w_2\,T_j^p \tag{21}
$$

where T_j^p denotes the temperature at the node j and time t^p.

Additionally

$$
T_j^{p+1} = f_{1\,j}^{p+1} + f_{2\,j}^{p+1} \tag{22}
$$

It should be pointed out that in numerical realization it is convenient to divide the algorithm into two steps:

- *collision step*: for each node the right-hand sides of equations (20) are calculated

$$
\begin{aligned}
f_{1\,j}^{p+1} &= \left(1-\frac{\Delta t}{\tau}\right) f_{1\,j}^{p} + \frac{\Delta t}{\tau} f_{1\,j}^{0\,p} + w_1\,\Delta t\,\frac{Q_j^p}{c}, \quad j = 0, 1, 2, \dots, n \\
f_{2\,j}^{p+1} &= \left(1-\frac{\Delta t}{\tau}\right) f_{2\,j}^{p} + \frac{\Delta t}{\tau} f_{2\,j}^{0\,p} + w_2\,\Delta t\,\frac{Q_j^p}{c}, \quad j = 0, 1, 2, \dots, n
\end{aligned}
\tag{23}
$$

- *streaming step*: obtained values are assigned to the adequate nodes (Fig. 2)

$$
\begin{aligned}
f_{1\,j+1}^{p+1} &= f_{1\,j}^{p+1}, \quad j = n-1, n-2, \dots, 0 \\
f_{2\,j-1}^{p+1} &= f_{2\,j}^{p+1}, \quad j = 1, 2, \dots, n
\end{aligned}
\tag{24}
$$

It is visible that two values are unknown, this means $f_{1\,0}^{p+1}$, $f_{2\,n}^{p+1}$ and these values are determined from the boundary conditions.

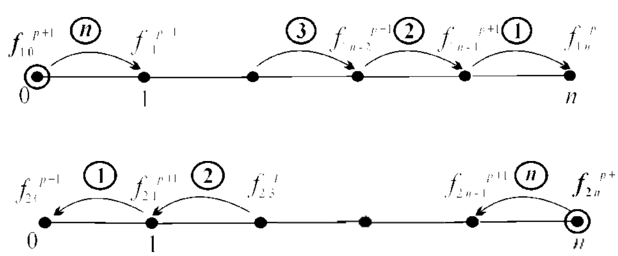

Fig. 2. Streaming process

For example, if for $x = 0$ and for $x = L$ the Dirichlet conditions: $T(x, 0) = T_w$, $T(x, L) = T_z$ are assumed then (c.f. formula (22))

$$f_{10}^{p+1} + f_{20}^{p+1} = T_w \rightarrow f_{10}^{p+1} = T_w - f_{20}^{p+1} \tag{25}$$

and

$$f_{1n}^{p+1} + f_{2n}^{p+1} = T_z \rightarrow f_{2n}^{p+1} = T_z - f_{1n}^{p+1} \tag{26}$$

In the case when for $x = L$ the Neumann condition: $-\lambda \partial T(x, t)/\partial x = q_b$ is accepted, one has

$$-\lambda \frac{T_n^{p+1} - T_{n-1}^{p+1}}{h} = q_b \rightarrow T_n^{p+1} = T_{n-1}^{p+1} - \frac{h}{\lambda} q_b \tag{27}$$

and next

$$f_{1n}^{p+1} + f_{2n}^{p+1} = f_{1n-1}^{p+1} + f_{2n-1}^{p+1} - \frac{h}{\lambda} q_b \rightarrow f_{2n}^{p+1} = f_{1n-1}^{p+1} + f_{2n-1}^{p+1} - f_{1n}^{p+1} - \frac{h}{\lambda} q_b \tag{28}$$

For Robin condition: $x = L: -\lambda \partial T(x, t)/\partial x = \alpha[T(x, t) - T_a]$ one has

$$-\lambda \frac{T_n^{p+1} - T_{n-1}^{p+1}}{h} = \alpha\left(T_n^{p+1} - T_a\right) \rightarrow T_n^{p+1} = \frac{\lambda}{\lambda + \alpha h} T_{n-1}^{p+1} + \frac{\alpha h}{\lambda + \alpha h} T_a \tag{29}$$

this means

$$\begin{gathered} f_{1n}^{p+1} + f_{2n}^{p+1} = \frac{\lambda}{\lambda + \alpha h}\left(f_{1n-1}^{p+1} + f_{2n-1}^{p+1}\right) + \frac{\alpha h}{\lambda + \alpha h} T_a \rightarrow f_{2n}^{p+1} = \\ \frac{\lambda}{\lambda + \alpha h}\left(f_{1n-1}^{p+1} + f_{2n-1}^{p+1}\right) - f_{1n}^{p+1} + \frac{\alpha h}{\lambda + \alpha h} T_a \end{gathered} \tag{30}$$

3. Finite difference method

To verify the LBM the same problem has been solved using the explicit scheme of the finite difference method (FDM) [6-8]. For 1D problem and internal node j the following approximation of equation (1) is used

$$c \frac{T_j^{p+1} - T_j^p}{\Delta t} = \lambda \frac{T_{j-1}^p - 2T_j^p + T_{j+1}^p}{h^2} + Q_j^p \tag{31}$$

where h is the constant mesh step, $T_{j-1}^{p} = T\left(x_{j-1}, t^{p}\right)$, $T_{j}^{p} = T\left(x_{j}, t^{p}\right)$ etc.
From equation (31) results that

$$T_j^{p+1} = \left(1 - \frac{2a\Delta t}{h^2}\right)T_j^p + \frac{a\Delta t}{h^2}\left(T_{j-1}^p + T_{j+1}^p\right) + \frac{Q_j^p}{c}\Delta t \tag{32}$$

The criterion of stability of this explicit differential scheme is following: $1-\left(2a\Delta t/h^2\right) \geq 0$ [7, 8].

4. Results of computations

The layer of thickness L = 0.05 m made of steel is considered. In computations the following input data are introduced: λ = 35 W/(mK), c = 4.875 MJ/(m^{3}K), source function $Q = 0$. Initial temperature equals to $T_p = 0$°C, mesh step: h = 0.00125 m (n = 40), time step: $\Delta t = 0.1$ s.

In the first example of computations the Dirichlet conditions in the form $T(0, t)$ = = 0°C and $T(L, t)$ = 100°C are assumed. In Figure 3 the temperature distributions obtained by LBM algorithm (symbols) and by FDM algorithm (solid lines) for different moments of time are shown.

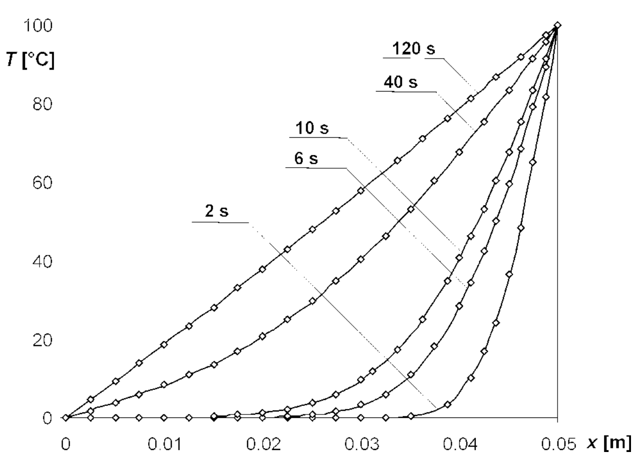

Fig. 3. LBM (symbols) and FDM (solid lines) solutions - example 1

Figure 4 illustrates the temperature distributions under the assumption that for $x = 0$: $T(0, t) = 150$°C (Dirichlet condition) and for $x = L$: $-\lambda\, \partial T(x, t)/\partial x$ = = $\alpha[T(x, t) - T_a]$ (Robin condition, α = 10 W/(m^{2}K), T_a = 20°C).

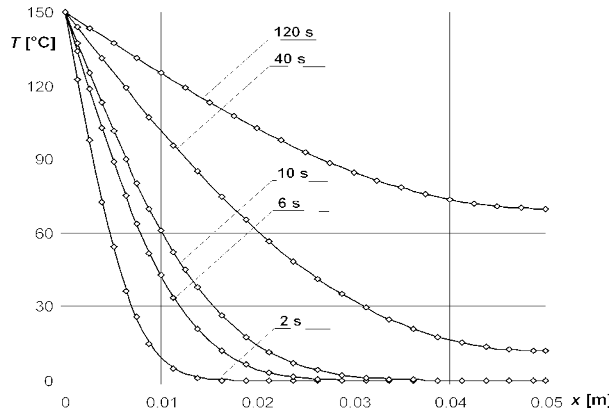

Fig. 4. LBM (symbols) and FDM (solid lines) solutions - example 2

Conclusions

The lattice Boltzmann method for the 1D Fourier equation supplemented by different boundary conditions and initial condition has been presented. The exemplary tasks have been solved both by the lattice Boltzmann method and by the explicit scheme of the finite different method. The good agreement of the solutions obtained has been observed.

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