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ON WEIGHTS WHICH ADMIT HARMONIC BERGMAN KERNEL AND MINIMAL SOLUTIONS OF LAPLACE'S EQUATION

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Abstract. In this paper we consider spaces of weight square-integrable and harmonic functions $L^2H(\Omega,\mu)$. Weights μ for which there exists reproducing kernel of $L^2H(\Omega,\mu)$ are named 'admissible weights' and such kernels are named 'harmonic Bergman kernels'. We prove that if only weight of integration is integrable in some negative power, then it is admissible. Next we construct a weight μ on the unit circle which is non-admissible and using Bell-Ligocka theorem we show that such weights exist for a large class of domains in $\mathbb{R}^2$. Later we conclude from the classical result of reproducing kernel Hilbert spaces theory that if the set $\{f \in L^2H(\Omega,\mu) | f(z) = c\}$ for admissible weight μ is non-empty, then there is exactly one element with minimal norm. Such an element in this paper is called 'a minimal (z,c)-solution in weight μ of Laplace's equation on Ω' and upper estimates for it are given.

1. Historic background

It is well-known that the Hilbert space of square-integrable and harmonic functions is a reproducing kernel Hilbert space. Both cases of real-valued and complex-valued harmonic functions were examined. We have a direct formula

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for the reproducing kernel of such a space on the unit ball in $\mathbb{R}^n$ (see e.g. [1]) and estimates for other domains (see e.g. [6], [10]). There is also a paper in which weighted harmonic Bergman kernel is considered (see [7]), but weights are very specific — they are some power of distances of points of a domain from the boundary.

Z. Pasternak Winiarski (see [9]) was the one who considered 'admissible weights' for the case of classical Bergman space of holomorphic functions.

Note also that usually term 'minimal solution of a differential equation' denotes different kind of extremal solutions (see e.g. [3], [12]) than in this paper.

2. Bergman space of harmonic functions

By domain in the whole paper we will understand an open, connected and non-empty subset of $\mathbb{R}^n$.

Let Ω be a bounded domain in $\mathbb{R}^N$. Let $\mu\colon \Omega \to \mathbb{R}$ be measurable and positive a.e. (such a function will be called *a weight*). By $L^2(\Omega, \mu)$ we will denote the set of (classes of) functions $f\colon \Omega \to \mathbb{R}$ square integrable in the sense

$$||f||_\mu := \int_\Omega |f(w)|^2 \mu(w) \mathrm{d}w < \infty.$$

By $L^2H(\Omega, \mu)$ we will denote the set of functions from $L^2(\Omega, \mu)$ which are harmonic on Ω and real valued. Such a space will be called *the harmonic Bergman space*. Inner product on it can be defined in the following way

$$\langle f|g\rangle_\mu := \int_\Omega f(w)g(w)\mu(w)\mathrm{d}w.$$

Sometimes we will simplify the above notation to $\langle f|g\rangle$ and $||f||$, when it will not cause misunderstanding.

Proposition 2.1. *$L^2H(\Omega, 1)$ is closed in $L^2(\Omega, 1)$ topology, i.e. it is a Hilbert space.*

Proof. Let D be Laplace's operator, h be any smooth function with compact support and $f_n \to f$ in $L^2(\Omega, 1)$ topology. Then $Df_n = 0$ and

$$0 = \langle Df_n|h\rangle = \langle f_n|D^*h\rangle = \langle f_n|Dh\rangle,$$

because the Laplace's operator is self-adjoint. Since $f_n \to f$ in $L^2(\Omega, 1)$ topology, f_n converges to f also in weak topology and

$$0 = \langle f|Dh\rangle.$$

Finally

$$0 = \langle Df|h\rangle.$$

Since h is an arbitrary element of a dense subspace of $L^2(\Omega, 1)$, we conclude that $Df = 0$. □

If weight μ is bounded from above and below by non-zero constants, then topology of $L^2H(\Omega, \mu)$ is the same as topology of $L^2H(\Omega, 1)$. In particular $L^2H(\Omega, \mu)$ is complete. Indeed, to see this we just need to write simple inequality:

$$\min_{w\in\Omega} \mu(w) \int_\Omega f(w)\mathrm{d}w \leq \int_\Omega f(w)\mu(w)\mathrm{d}w \leq \max_{w\in\Omega} \mu(w) \int_\Omega f(w)\mathrm{d}w,$$

which holds for function f which is non-negative for almost every point of Ω. We can also prove something more general:

PROPOSITION 2.2. *Let μ be a weight on Ω, such that:*

(CB) *for any compact set $X \subset \Omega$ there exists C_X such that for any $z \in X$ and any $f \in L^2H(\Omega, \mu)$ we have*

$$|f(z)| \leq C_X ||f||_\mu.$$

Then $L^2H(\Omega, \mu)$ is a closed subspace of $L^2(\Omega, \mu)$.

PROOF. If $f_n \to f$ in $L^2H(\Omega, \mu)$ topology, then by condition (CB) $f_n \to f$ also locally uniformly on Ω. Moreover

$$\int_X |f(w)|^2\mathrm{d}w \leq L(X) \sup_{z\in X} |f(z)|^2,$$

where $L(X)$ denotes Lebesgue measure of X. Therefore if $f_n \to f$ locally uniformly, then also $f_n \to f$ in $L^2H(X, 1)$ topology. And $L^2H(X, 1)$ by Proposition 2.1 is complete. □

A function $K \colon \Omega \times \Omega \to \mathbb{R}$, such that

(i) $K(z, \cdot) \in L^2H(\Omega, \mu)$ for any $z \in \Omega$;

(ii) $\langle f(\cdot)|K(z,\cdot)\rangle_\mu = f(z)$ for any $z \in \Omega$ and any $f \in L^2H(\Omega,\mu)$ (*reproducing property*);

will be called *the harmonic Bergman kernel* of space $L^2H(\Omega,\mu)$.

Note that not each Hilbert space of functions is equipped with reproducing kernel. Example of (weighted) classical Bergman space of holomorphic functions without corresponding reproducing kernel was given by Z. Pasternak-Winiarski in [9]. However, as in the case of classical Bergman space:

THEOREM 2.1. *The following conditions are equivalent:*

(i) *there exists harmonic Bergman kernel of* $L^2H(\Omega,\mu)$*;*

(ii) *condition (CB) holds.*

PROOF. (i) ⇒ (ii) By the reproducing property and Cauchy inequality

$$|f(z)| = |\langle f(\cdot)|K(z,\cdot)\rangle_\mu| \leq ||f||_\mu ||K(z,\cdot)||_\mu \leq \sqrt{K(z,z)}||f||_\mu.$$

(Note that

$$||K(z,\cdot)||_\mu^2 = \langle K(z,\cdot)|K(z,\cdot)\rangle = K(z,z)$$

by the reproducing property.) We can take

$$C_X := \max_{z\in X} \sqrt{K(z,z)}$$

for any compact $X \subset \Omega$, which is finite by general theory of reproducing kernels.

(ii) ⇒ (i) Condition (ii) implies that functionals of point evaluation, i.e. functionals

$$E_z\colon L^2H(\Omega,\mu) \ni f \mapsto f(z) \in \mathbb{R}$$

are continuous. If functionals of point evaluation are continuous, then by the Riesz representation theorem for any $z \in \Omega$ there exists $e_z \in L^2H(\Omega,\mu)$, such that

$$\langle f|e_z\rangle = f(z)$$

for any $f \in L^2H(\Omega,\mu)$. Function K defined in the following way

$$K(z,w) := e_z(w)$$

is the harmonic Bergman kernel of considered space. □

Weight μ for which condition (CB) is satisfied will be called *an admissible weight*.

2.1. Sufficient condition for a weight to be admissible

In the case of the classical Bergman space of holomorphic functions (see [9]), if only weight is integrable in some negative power, then it is admissible. The following sufficient condition for existence of harmonic Bergman kernel holds:

THEOREM 2.2. *Let μ be a weight on Ω, such that there exists $a \geq 1$, for which*

$$\int_\Omega \frac{1}{\mu^a(w)} \mathrm{d}w < \infty.$$

Then μ is admissible.

The difference comes from the fact that if f is holomorphic, then $|f|^p$ is subharmonic for any $p > 0$, but if f is (real) harmonic, then $|f|^p$ is subharmonic for any $p \geq 1$ (see [8]).

PROOF. Let $z \in \Omega$ be fixed. Let $r > 0$ be sufficiently small for a ball $B(z,r) := \{w \in \mathbb{R}^N | |w - z| < r\}$ to lie in Ω. Let $p := \frac{1+a}{a}$ and $q := 1 + a$. Let now $f \in L^2H(\Omega, \mu)$. Then

$$\frac{2}{p} \geq 1$$

and by the mean value theorem for subharmonic functions we have

$$|f(z)|^{\frac{2}{p}} \leq \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}} r^n} \int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mathrm{d}w.$$

(Note that the volume of n-dimensional ball of radius r is equal to

$$\frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}.$$

It is a classical result. See e.g. [4] for more details.)

Of course

$$\int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mathrm{d}w = \int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mu(w)^{\frac{1}{p}} \mu(w)^{-\frac{1}{p}} \mathrm{d}w.$$

Since $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we may use Hölder's inequality:

$$\int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mu(w)^{\frac{1}{p}} \mu(w)^{-\frac{1}{p}} \mathrm{d}w \leq \left(\int_{B(z,r)} |f(w)|^2 \mu(w) \mathrm{d}w \right)^{\frac{1}{p}} \left(\int_{B(z,r)} \mu(w)^{-\frac{q}{p}} \mathrm{d}w \right)^{\frac{1}{q}}.$$

So we have

$$|f(z)| \leq \left(\int_{B(z,r)} \mu(w)^{-\frac{q}{p}} \mathrm{d}w \right)^{\frac{p}{2q}} \left(\frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}} r^n} \right)^{\frac{p}{2}} ||f||_{\mu}.$$

Finally

$$|f(z)| \leq \left(\int_{B(z,r)} \mu(w)^{-a} \mathrm{d}w \right)^{\frac{1}{2a}} \left(\frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}} r^n} \right)^{\frac{1+a}{2a}} ||f||_{\mu}. \tag{2.1}$$

As we can see, constant C_z in the proof depends only on the distance of point z to the boundary. Since μ^{-a} is integrable on whole Ω, we conclude that in fact for any compact set $X \subset \Omega$ we can fix C_X, such that condition (CB) holds. □

2.2. Example of a weight which is not admissible

An example of a weight for which there is no reproducing kernel of corresponding classical (weighted) Bergman space was found by Z. Pasternak-Winiarski (see [9]). Here we will use a similar idea to give an example of non-admissible weight for harmonic Bergman space. We will need the following theorem by Runge (see [11] for more details):

Theorem 2.3. *Let $X \subset \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus X$ is connected. Let $f\colon X \to \mathbb{C}$ be continuous on X and holomorphic on the interior of X. Then f is a uniform limit of a sequence of holomorphic polynomials on X.*

Let Ω be the unit disk in $\mathbb{R}^2$. Let

$$A_n := \{(x,y) \in \mathbb{R}^2 : ||(x,y)|| < 2^{-n}\} \cup \{(x,y) \in \mathbb{R}^2 : |y| < 2^{-n} \wedge 0 < x < 1\},$$

where $||\cdot||$ is the classical norm on $\mathbb{R}^2$. Let

$$M_n := (\overline{\Omega \setminus A_n}) \cup \overline{A_{n+1}}.$$

Now let $f_n\colon M_n \to \mathbb{R}^2$ be defined in the following way

$$f_n(x,y) := \begin{cases} 1+\frac{1}{n} & \text{for} \quad (x,y) \in \overline{A_{n+1}}, \\ 0 & \text{for} \quad (x,y) \in \overline{\Omega \setminus A_n}. \end{cases}$$

By Theorem 2.3 there exist holomorphic polynomials G_n such that

$$|G_n(x,y) - f_n(x,y)| < \frac{1}{n}$$

for any $(x,y) \in M_n$. Bearing in mind that a sequence of holomorphic functions is convergent if and only if its real and imaginary part are convergent and imaginary part of f_n is zero, we conclude that in fact there exist harmonic polynomials g_n such that

$$|g_n(x,y) - f_n(x,y)| < \frac{1}{n}$$

for any $(x,y) \in M_n$. It implies that $|g_n(x,y)| < \frac{1}{n}$ for $(x,y) \in \overline{\Omega \setminus A_n}$ and $1 < |g_n(x,y)| < 1 + \frac{2}{n}$ for $(x,y) \in \overline{A_{n+1}}$. Now let us define polynomials:

$$h_n(x,y) := \frac{g_n(x,y)}{g_n(0,0)}.$$

Since $|g_n(0,0)| > 1$, h_n is well-defined. Moreover

$$\left(1+\frac{2}{n}\right)^{-1} < |h_n(x,y)| < 1 + \frac{2}{n}$$

on $\overline{A_{n+1}}$ and

$$|h_n(x,y)| < \frac{1}{n}$$

on $\overline{\Omega \setminus A_n}$. Now let us denote

$$D_n := \Omega \cap \overline{A_n}.$$

Then we may define a weight:

$$\mu(x,y) := \begin{cases} 1 & \text{if} \quad (x,y) \in \Omega \setminus D_1, \\ 0 & \text{if} \quad x \in [0,1) \wedge y = 0, \\ \min\left\{1, \frac{1}{|h_n(x,y)|^2}\right\} & \text{if} \quad (x,y) \in D_n \setminus D_{n+1}. \end{cases} \tag{2.2}$$

Since μ is bounded from above (by 1), $h_n \in L^2H(\Omega,\mu)$ for any $n \in \mathbb{N}$, as harmonic polynomials. It is easy to show that

$$|h_n(x,y)|^2\mu(x,y) < 9$$

and

$$\lim_{n\to\infty} |h_n(x,y)|^2\mu(x,y) = 0.$$

Therefore, by the Lebesgue Majorized Convergence Theorem we have

$$\int_\Omega \lim_{n\to\infty} |h_n(x,y)|^2\mu(x,y)\mathrm{d}w = \lim_{n\to\infty} \int_\Omega |h_n(x,y)|^2\mu(x,y)\mathrm{d}w = 0.$$

By its own definition, $|h_n(0,0)| = 1$ for any $n \in \mathbb{N}$, but $||h_n||_\mu \to 0$. It means that functional of point evaluation $L^2H(\Omega,\mu) \ni f \mapsto f(0,0) \in \mathbb{R}$ is not continuous and by Theorem 2.1 harmonic Bergman kernel of $L^2H(\Omega,\mu)$ does not exist.

2.3. Weights and biholomorphisms

Here we will need the following theorem:

THEOREM 2.4. *Let Ω_1, Ω_2 be domains in $\mathbb{C}^N$ of one of the following types:*

Type 1: *smooth bounded pseudoconvex domain with the real analytic boundary;*

Type 2: *smooth bounded strictly pseudoconvex domain and (more generally);*

Type 3: *smooth bounded domain for which a $\overline{\partial}$-operator exists and satisfies subelliptic estimates.*

Then any biholomorphic mapping between Ω_1 and Ω_2 extends smoothly to the boundary.

This theorem was proved by S. Bell and E. Ligocka in [2]. Note that each (geometrically) convex domain is pseudoconvex and moreover in $\mathbb{C}^1$ each domain is pseudoconvex. (See [5] or [8] for more details.)

In the following two theorems we are going to use the same symbol for biholomorphism and its smooth prolongation to the boundary, if it exists, which should not be misleading.

THEOREM 2.5. *Let* $\Omega_1, \Omega_2 \subset \mathbb{C}^1 = \mathbb{R}^2$ *be biholomorphic domains of* Type 1, 2 *or* 3 *from Theorem* 2.4. *Let* $\Phi\colon \Omega_1 \to \Omega_2$ *be a biholomorphism and* μ *be a weight on* Ω_2. *Then*

(i) *for any* g *measurable and non-negative almost everywhere we have:*

$$\int_{\Omega_2} g(w)\mu(w)\mathrm{d}w < \infty \Leftrightarrow \int_{\Omega_1} (g\circ\Phi)(w)(\mu\circ\Phi)(w)\mathrm{d}w < \infty;$$

in particular, $h \in L^2H(\Omega_2,\mu)$ *if and only if* $h\circ\Phi \in L^2H(\Omega_1, \mu\circ\Phi)$.

(ii) μ *is admissible on* Ω_2 *if and only if* $\mu\circ\Phi$ *is admissible on* Ω_1.

PROOF. (i) First let us recall that

$$\int_{\Omega_1} (g\circ\Phi)(w)(\mu\circ\Phi)(w)|\det J_{\mathbb{C}}\Phi(w)|^2\mathrm{d}w = \int_{\Omega_2} g(w)\mu(w)\mathrm{d}w. \tag{2.3}$$

By the fact that $u := |\det J_{\mathbb{C}}\Phi|$ is a smooth function on compact set $\overline{\Omega_1}$ (see Theorem 2.4), we have

$$\begin{aligned} C_1\int_{\Omega_1} (g\circ\Phi)(w)(\mu\circ\Phi)(w)\mathrm{d}w &\le \int_{\Omega_1} (g\circ\Phi)(w)(\mu\circ\Phi)(w)|\det J_{\mathbb{C}}\Phi|^2\mathrm{d}w \\ &\le C_2\int_{\Omega_1} (g\circ\Phi)(w)(\mu\circ\Phi)(w)\mathrm{d}w, \end{aligned}$$

where $C_1 := \min_{w\in\overline{\Omega}} u(w) > 0$ and $C_2 := \max_{w\in\overline{\Omega}} u(w)$. By (2.3) we can change integral in the middle to get:

$$\begin{aligned} C_1\int_{\Omega_1} (g\circ\Phi)(w)(\mu\circ\Phi)(w)\mathrm{d}w &\le \int_{\Omega_2} g(w)\mu(w)\mathrm{d}w \\ &\le C_2\int_{\Omega_1} (g\circ\Phi)(w)(\mu\circ\Phi)(w)\mathrm{d}w. \end{aligned} \tag{2.4}$$

If the integral on the right hand side is finite, then the integral in the middle must be also finite and if the integral in the middle is finite, then the integral on the left hand side must be also finite.

For the proof of the second part of (i) we just recall that a composition of harmonic and holomorphic function is a harmonic function.

(ii) Since Φ is biholomorphism, we need only to show implication in one direction.

If μ is admissible on Ω_2, then for any compact set $X \subset \Omega_2$, $w \in X$ and any $f \in L^2H(\Omega_2, \mu)$ we have

$$|f(w)| \le C_X \sqrt{\int_{\Omega_2} |f(w)|^2 \mu(w) \mathrm{d}w}. \tag{2.5}$$

By using (2.4) for inequality (2.5) we gain

$$|(f \circ \Phi)(\tilde{w})| \le C_X \sqrt{C_2} \sqrt{\int_{\Omega_1} |f \circ \Phi(w)|^2 (\mu \circ \Phi)(w) \mathrm{d}w},$$

for $\Omega_1 \supset Y := \Phi^{-1}(X)$, $\tilde{w} := \Phi^{-1}(w) \in Y$, so (CB) is satisfied for $C_Y := C_X \sqrt{C_2}$. □

Corollary 2.1. *For any simply-connected bounded domain Ω in $\mathbb{R}^2 = \mathbb{C}$ which is of* Type 1-3 *there exists a non-admissible weight on Ω.*

Proof. By the Riemann Mapping Theorem there exists biholomorphism $\Phi\colon \Omega \to K(0,1)$. By Theorem 2.4 Φ has a smooth prolongation to $\partial\Omega$. By Theorem 2.5 the weight $\mu \circ \Phi$, where μ is a weight constructed in (2.2), is non-admissible weight on Ω. □

3. Reproducing kernel Hilbert space and minimal norm property

The content of this section is true for general reproducing kernel Hilbert spaces and well-known. We decided, however, to give details for completness. In the whole section we will assume that μ is admissible weight, without further reminding.

Theorem 3.1. *If $K_\mu(z,z) \neq 0$, then*

$$k_z(\cdot) := \frac{\overline{K_\mu(z,\cdot)}}{K_\mu(z,z)}$$

is the only element of $\mathcal{H}$ with the following properties:

(i) $k_z(z) = 1$;

(ii) *if* $m_z \in \mathcal{H}$, $m_z(z) = 1$ *and* $||m_z|| \leq ||k_z||$, *then* $m_z = k_z$. *Moreover*

$$||k_z|| = \frac{1}{\sqrt{K_\mu(z,z)}}.$$

Proof. By the reproducing property and the Cauchy inequality for any $f \in \mathcal{H}$, $z \in U$ we have

$$|f(z)| = |\langle f(w)|K_\mu(z,w)\rangle| \leq ||f(w)|| \cdot ||K_\mu(z,\cdot)||,$$

i.e.

$$|f(z)| \leq \sqrt{K_\mu(z,z)}||f||. \tag{3.1}$$

Moreover $\sqrt{K_\mu(z,z)}$ is the smallest possible constant for which inequality (3.1) holds. Indeed, let $E_z \colon L^2H(\Omega,\mu) \ni f \mapsto f(z) \in \mathbb{C}$ be functional of point evaluation. By the Riesz correspondence theorem,

$$||E_z||^* = |\overline{K_\mu(z,\cdot)}||,$$

but

$$||\overline{K_\mu(z,\cdot)}|| = \sqrt{K_\mu(z,z)}.$$

At once $||\overline{K_\mu(z,\cdot)}||_\mu$ is by definition the smallest constant for which inequality (3.1) holds.

Now we have

$$\frac{1}{\sqrt{K_\mu(z,z)}} \leq \frac{||f||}{|f(z)|} = \left|\left|\frac{f}{f(z)}\right|\right|.$$

But

$$\left|\left|\frac{\overline{K_\mu(z,w)}}{K_\mu(z,z)}\right|\right|^2 = \frac{1}{K_\mu(z,z)}$$

by the reproducing property. To end the proof we need only to show that, if $||m_z|| = ||k_z||$, then $m_z = k_z$. Note that for $f_z := \frac{1}{2}(m_z + k_z)$ we have $f_z(z) = 1$ and

$$||f_z|| = \left|\left|\frac{1}{2}(m_z + k_z)\right|\right| \leq \frac{1}{2}(||m_z|| + ||k_z||) = ||k_z||.$$

On the other hand we showed above that

$$||f_z|| \geq ||k_z||,$$

so $||f_z|| = ||k_z||$. Since in our case the triangle inequality is in fact an equality and each Hilbert space is strictly convex, there exists $\alpha \in \mathbb{C}$, such that $m_z = \alpha k_z$. Thus

$$\left|\left|\frac{1}{2}(m_z + k_z)\right|\right| = \frac{1}{2}(\alpha + 1)||k_z||.$$

Since

$$||f_z|| = ||k_z||,$$

we see that $\alpha = 1$ and in conclusion $m_z = k_z$. □

In fact, something more general is true.

PROPOSITION 3.1. *Let $c \in \mathbb{R}$. In the set $V_{z,c} = \{f \in L^2H(\Omega, \mu) | f(z) = c\}$, if non-empty, there is exactly one element with minimal norm. Such an element will be called a (z,c)-minimal solution of Laplace's equation in weight μ on Ω.*

PROOF. For $c = 1$ it is just a consequence of Theorem 3.1. It is obvious that for $c \neq 0$, the linear operator

$$Af := cf$$

is a bijection between $V_{z,1}$ and $V^{\mu}_{z,c}$, and

$$||Af||_\mu = |c| \cdot ||f||_\mu.$$

Therefore in $V^{\mu}_{z,\ c}$ there is exactly one element f_c with minimal norm and

$$f_c = cf_1,$$

where f_1 is the unique element of $V_{z,1}$ with minimal norm.

Now let us consider the case $c = 0$. Of course zero is the only element of $V^{\mu}_{z,\ 0}$ with minimal norm. □

Now we will investigate the case in which $f(z) = 0$ for each $f \in \mathcal{H}$ and some $z \in \Omega$.

THEOREM 3.2. *The following conditons are equivalent for a point $z \in \Omega$:*
(i) $f(z) = 0$ *for any* $f \in L^2H(\Omega, \mu)$;
(ii) $K_\mu(z, z) = 0$;
(iii) $K_\mu(z, \cdot) \equiv 0$.

Proof. (i) $\Rightarrow$ (ii) If for some $z \in U$ we have $f(z) = 0$ for any $f \in \mathcal{H}$, then in particular for $g(\cdot) = \overline{K_\mu(z,\cdot)}$ we have $g(z) = 0$.

(ii) $\Rightarrow$ (iii) Because

$$||K_\mu(z,\cdot)||^2 = K_\mu(z,z) = 0$$

and integrated function is continuous and non-negative, $K_\mu(z,\cdot) \equiv 0$ on U.

(iii)$\Rightarrow$ (i) By the reproducing property, for any $f \in L^2H(\Omega,\mu)$, we have

$$f(z) = \langle f(w)|K_\mu(z,w)\rangle = \langle f|0\rangle = 0. \qquad \square$$

4. Upper estimates for minimal solutions of Laplace's equation

By Proposition 3.1 in the set of weight square-integrable solutions of Laplace's equation which take value equal to c at some given point, if not empty, there exists exactly one element with minimal norm.

Theorem 4.1. *Let μ be a weight on $\Omega \subset \mathbb{R}^n$, such that*

$$\int_\Omega \frac{1}{\mu(x)} \mathrm{d}x < \infty$$

and

$$\int_\Omega \mu(x) \mathrm{d}x < \infty.$$

Let f denote minimal (z,c)-solution in weight μ of Laplace's equation on Ω. Then

$$|f(w)| \le |c| \sqrt{\int_\Omega \mu(x)\mathrm{d}x} \sqrt{\int_\Omega \frac{1}{\mu(x)}\mathrm{d}x} \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}\delta(w)^n},$$

where $\delta(w)$ denotes the distance of w to the boundary of Ω. In fact

$$|f(w)| \le |c| \sqrt{\int_\Omega \mu(x)\mathrm{d}x} \sqrt{\int_{B(w,\varepsilon\delta(w))} \frac{1}{\mu(x)}\mathrm{d}x} \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}(\varepsilon\delta(w))^n}$$

for any $0 < \varepsilon \le 1$.

In particular, if μ is equal to 1 almost everywhere, then

$$|f(w)| \leq |c| L(\Omega) \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}} \delta(w)^n},$$

where $L(\Omega)$ denotes the Lebesgue measure of Ω.

Note that the estimate does not depend on point z.

Proof. First let us consider the situation when $c = 1$. By Theorem 3.1

$$|f(w)| = \left| \frac{K_\mu(z,w)}{K_\mu(z,z)} \right|.$$

By the Cauchy–Schwarz inequality

$$|K_\mu(z,w)| = |\langle K_\mu(z,\cdot)|K_\mu(w,\cdot)\rangle| \leq ||K_\mu(z,z)|| \cdot ||K_\mu(w,w)||$$
$$= \sqrt{K_\mu(z,z)}\sqrt{K_\mu(w,w)}$$

and in consequence

$$|f(w)| \leq \frac{\sqrt{K_\mu(w,w)}}{\sqrt{K_\mu(z,z)}}.$$

(Remember that $||K_\mu(z,\cdot)||^2 = \langle K_\mu(z,\cdot)|K_\mu(z,\cdot)\rangle = K_\mu(z,z)$ by the reproducing property.)

By the Riesz representation theorem $||E_w||^* = ||K_\mu(w,\cdot)|| = \sqrt{K_\mu(w,w)}$. On the other hand $||E_w||^*$ is the smallest real number C_w for which inequality

$$|f(w)| \leq C_w ||f||_\mu$$

holds for any $f \in L^2H(\Omega,\mu)$. Combining this with inequality (2.1) for $a = 1$, we get

$$\sqrt{K_\mu(w,w)} \leq \sqrt{\int_{B(w,r)} \frac{1}{\mu(x)} \mathrm{d}x \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}} r^n}}$$

for $r \leq \delta(w)$. Moreover by the reproducing property and the Cauchy–Schwarz inequality

$$1 = |\langle 1|K_\mu(z,\cdot)\rangle| \leq ||1|| \cdot ||K_\mu(z,\cdot)|| = \sqrt{\int_\Omega \mu(x)\mathrm{d}x}\sqrt{K_\mu(z,z)}.$$

Therefore

$$\frac{1}{\sqrt{K_\mu(z,z)}} \leq \sqrt{\int_\Omega \mu(x)\mathrm{d}x}.$$

If $c \neq 1$, $c \neq 0$, then a minimal (z,c)-solution in weight μ is equal to minimal $(z,1)$-solution in weight μ multiplied by c, as in the proof of theorems from the previous section.

If $c = 0$, then inequality from the theorem is trivial. □

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