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AN EXTENSION OF THE ABEL–LIOUVILLE IDENTITY

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Abstract. In this note, we present an extension of the celebrated Abel–Liouville identity in terms of noncommutative complete Bell polynomials for generalized Wronskians. We also characterize the range equivalence of n-dimensional vector-valued functions in the subclass of n-times differentiable functions with a nonvanishing Wronskian.

1. Introduction

Throughout this paper let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of real and the sets of positive and nonnegative integers, respectively, and let I stand for a nonempty open real interval.

For an n-dimensional vector-valued $(n-1)$-times continuously differentiable function $f\colon I \to \mathbb{R}^n$, its *Wronskian* $W_f\colon I \to \mathbb{R}$ is defined by

$$W_f := \left| f^{(n-1)} \quad \dots \quad f' \quad f \right|.$$

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Here we usually interpret the elements of $\mathbb{R}^n$ as column vectors. In the sequel, the standard inner product on $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$.

Consider now the nth-order homogeneous linear differential equation

$$y^{(n)} = a_1 y^{(n-1)} + \cdots + a_n y, \tag{1}$$

where $a_1, \ldots, a_n \colon I \to \mathbb{R}$ are continuous functions. By the classical *Abel–Liouville identity* (cf. [4]), if $f \colon I \to \mathbb{R}^n$ is a fundamental system of solutions of (1), then W_f does not vanish on I and

$$W_f' = a_1 W_f.$$

For a sufficiently smooth function $f \colon I \to \mathbb{R}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, we introduce now the *generalized Wronskian* $W_f^k \colon I \to \mathbb{R}$ by

$$W_f^k := \left| f^{(k_1)} \quad \ldots \quad f^{(k_n)} \right|.$$

One can easily see that, with this notation, we have

$$W_f = W_f^{(n-1,n-2,\ldots,0)} \quad \text{and} \quad W_f' = W_f^{(n,n-2,\ldots,0)}.$$

Therefore, the Abel–Liouville identity can be rewritten as

$$W_f^{(n,n-2,\ldots,0)} = a_1 W_f^{(n-1,n-2,\ldots,0)}. \tag{2}$$

One of the main goals of this short paper is to establish a formula for W_f^k in terms of the coefficients of differential equation (1). Another goal is to introduce the range equivalence for n-dimensional vector-valued functions and to characterize this equivalence relation in the subclass of n-times differentiable functions with a nonvanishing Wronskian.

2. Main results

For the description of our main result, we recall the notion of *noncommutative complete Bell polynomials*, which was introduced by Schimming and Rida ([3]). Let $\mathbb{R}^{n \times n}$ denote the ring of $n \times n$ matrices with real entries and let $\mathbb{I}_n$ denote the $n \times n$ unit matrix. Now define $B_m\colon (\mathbb{R}^{n\times n})^m \to \mathbb{R}^{n\times n}$ by the following recursive formula

$$B_0 := \mathbb{I}_n, \quad B_{m+1}(X_1,\dots,X_{m+1}) := \sum_{j=0}^{m} \binom{m}{j} B_j(X_1,\dots,X_j)X_{m+1-j}.$$

The notion of *complete Bell polynomials* in the commutative setting (i.e., when $n = 1$) was introduced by Bell ([1], [2]). One can easily compute the first few Bell polynomials as follows:

$$\begin{aligned}
B_1(X_1) &= X_1,\\
B_2(X_1,X_2) &= X_1^2 + X_2,\\
B_3(X_1,X_2,X_3) &= X_1^3 + 2X_1X_2 + X_2X_1 + X_3,\\
B_4(X_1,X_2,X_3,X_4) &= X_1^4 + 3X_1^2X_2 + 2X_1X_2X_1 + 3X_1X_3 + 3X_2^2\\
&\quad + X_2X_1^2 + X_3X_1 + X_4,\\
B_5(X_1,X_2,X_3,X_4,X_5) &= X_1^5 + 4X_1^3X_2 + 3X_1^2X_2X_1 + 6X_1^2X_3 + 8X_1X_2^2\\
&\quad + 2X_1X_2X_1^2 + 3X_1X_3X_1 + 4X_1X_4 + 3X_2^2X_1\\
&\quad + X_2X_1^3 + 4X_2X_1X_2 + 6X_2X_3 + X_3X_1^2\\
&\quad + 4X_3X_2 + X_4X_1 + X_5.
\end{aligned}$$

The statement of the next basic lemma was proved in the paper [3].

Lemma 1. *For every $j \in \mathbb{N}_0$, and j-times differentiable matrix-valued function $X\colon I \to \mathbb{R}^{n\times n}$,*

$$B_{j+1}\big(X,\dots,X^{(j)}\big) = XB_j\big(X,\dots,X^{(j-1)}\big) + \Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)'.$$

Lemma 2. *Let $n,m \in \mathbb{N}$, let $X\colon I \to \mathbb{R}^{n\times n}$ be an $(m-1)$-times continuously differentiable function and $Y\colon I \to \mathbb{R}^{n\times n}$ be a differentiable function such that*

$$Y' = YX \tag{3}$$

holds on I. Then Y is m-times continuously differentiable and

$$Y^{(j)} = YB_j\big(X,\dots,X^{(j-1)}\big) \quad (j \in \{0,\dots,m\}). \tag{4}$$

Proof. If $m=1$, then X is continuous, hence the continuity of Y and (3) imply that Y is continuously differentiable. If $m>1$, then using (3), a simple inductive argument shows that Y is m-times continuously differentiable.

The equality (4) is trivial if $j=0$, because $B_0=\mathbb{I}_n$. For $j=1$, the equality (4) is equivalent to (3). Now assume that (4) has been verified for some j with $1\le j<m$. Then, using (3) and Lemma 1, we get

$$\begin{aligned}
Y^{(j+1)} &= \big(Y^{(j)}\big)' = \Big(YB_j\big(X,\dots,X^{(j-1)}\big)\Big)' \\
&= Y'B_j\big(X,\dots,X^{(j-1)}\big) + Y\Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)' \\
&= Y\Big[XB_j\big(X,\dots,X^{(j-1)}\big) + \Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)'\Big] \\
&= YB_{j+1}\big(X,\dots,X^{(j)}\big).
\end{aligned}$$

This proves the assertion for $j+1$. □

In what follows, let $e_1,\dots,e_n$ denote the elements of the standard basis in $\mathbb{R}^n$.

Corollary 3. *Let $n,m\in\mathbb{N}$, let $a=(a_1,\dots,a_n)\colon I\to\mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f\colon I\to\mathbb{R}^n$ be a fundamental system of solutions of the differential equation* (1)*. Let the matrix-valued functions $X_a\colon I\to\mathbb{R}^{n\times n}$ and $Y_f\colon I\to\mathbb{R}^{n\times n}$ be defined by*

$$X_a := \begin{pmatrix} a & e_1 & \dots & e_{n-1}\end{pmatrix} \quad \text{and} \quad Y_f := \begin{pmatrix} f^{(n-1)} & \dots & f' & f\end{pmatrix}. \tag{5}$$

Then Y_f is m-times continuously differentiable and

$$Y_f^{(j)} = Y_fB_j\big(X_a,\dots,X_a^{(j-1)}\big) \quad (j\in\{0,\dots,m\}).$$

Proof. The function f satisfies the differential equation (1), therefore $f^{(n)}=Y_f\cdot a$. On the other hand, $f^{(n-i)}=Y_f\cdot e_i$ holds for $i\in\{1,\dots,n-1\}$. These equalities imply that

$$\begin{aligned}
Y_f' &= \begin{pmatrix} f^{(n)} & f^{(n-1)} & \dots & f'\end{pmatrix} \\
&= \begin{pmatrix} Y_f\cdot a & Y_f\cdot e_1 & \dots & Y_f\cdot e_{n-1}\end{pmatrix} = Y_fX_a.
\end{aligned}$$

Therefore, equation (3) holds with $Y:=Y_f$ and $X:=X_a$, consequently, the statement is a consequence of Lemma 2. □

Using the above corollary, we can easily establish a formula for the computation of the generalized Wronskian W_f^k.

Theorem 4. *Let $n, m \in \mathbb{N}$, let $a = (a_1, \dots, a_n)\colon I \to \mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f\colon I \to \mathbb{R}^n$ be a fundamental system of solutions of the differential equation* (1). *Let the matrix-valued functions $X_a\colon I \to \mathbb{R}^{n\times n}$ be defined by* (5). *Then, for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $\max(k_1, \dots, k_n) \le m + n - 1$,*

$$W_f^k = W_f \Big| B_{\ell_1}\big(X_a, \dots, X_a^{(\ell_1-1)}\big)e_{n+\ell_1-k_1} \quad \dots \quad B_{\ell_n}\big(X_a, \dots, X_a^{(\ell_n-1)}\big)e_{n+\ell_n-k_n}\Big|, \tag{6}$$

where, for $i \in \{1, \dots, n\}$, $\ell_i := (k_i - n + 1)^+ := \max(k_i - n + 1, 0)$.

Proof. Define the matrix valued function $Y_f\colon I \to \mathbb{R}^{n\times n}$ by (5) and observe that, in view of Corollary 3, for all $\ell \in \{0, \dots, m + n - 1\}$ and $i \in \{(\ell - n + 1)^+, \dots, \min(\ell, m)\}$, we have that

$$f^{(\ell)} = Y_f^{(i)} e_{n+i-\ell} = Y_f B_i\big(X_a, \dots, X_a^{(i-1)}\big)e_{n+i-\ell}.$$

By taking the smallest possible value for i in the above formula, we get

$$f^{(\ell)} = Y_f B_{(\ell-n+1)^+}\big(X_a, \dots, X_a^{((\ell-n+1)^+-1)}\big)e_{n+(\ell-n+1)^+-\ell}.$$

Applying this equality for $\ell \in \{k_1, \dots, k_n\}$, we obtain

$$\big(f^{(k_1)} \quad \dots \quad f^{(k_n)}\big) = Y_f\Big(B_{\ell_1}\big(X_a, \dots, X_a^{(\ell_1-1)}\big)e_{n+\ell_1-k_1} \quad \dots \quad B_{\ell_n}\big(X_a, \dots, X_a^{(\ell_n-1)}\big)e_{n+\ell_n-k_n}\Big).$$

Now taking the determinant side by side and using the product rule for determinants, the equality (6) follows. □

In the subsequent corollary, we consider the case when $\ell_i = 0$ for $i \in \{2, \dots, n\}$. In this particular setting, the determinant on the left hand side of (6) can easily be computed.

Corollary 5. *Let $n, m \in \mathbb{N}$, let $a = (a_1, \dots, a_n)\colon I \to \mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f\colon I \to \mathbb{R}^n$ be a fundamental system of solutions of the differential equation* (1). *Let the matrix-valued* *functions $X_a\colon I \to \mathbb{R}^{n\times n}$ be defined by* (5) *and let $d \in \{0,...,m-1\}$ and $j \in \{0,\dots,n-1\}$. Then*

$$W_f^{(n+d,n-1,\dots,j+1,j-1,\dots,0)} = (-1)^{n-j-1} W_f \langle B_{d+1}\bigl(X_a,\dots,X_a^{(d)}\bigr)e_1, e_{n-j}\rangle. \tag{7}$$

If $d = 0$ and $j = n-1$, then this equality reduces to the Abel–Liouville identity (2)*. More generally, for $d = 0,1,2$, we get the following formulas:*

$$\begin{aligned}
W_f^{(n,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f a_{n-j},\\
W_f^{(n+1,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f (a_1 a_{n-j} + a_{n-j+1} + a'_{n-j}),\\
W_f^{(n+2,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f (a_1^2 a_{n-j} + a_1 a_{n-j+1} + a_2 a_{n-j}\\
&\quad + a_{n-j+2} + a_1 a'_{n-j} + 2a'_1 a_{n-j} + 2a'_{n-j+1} + a''_{n-j}).
\end{aligned} \tag{8}$$

(Here we define $a_{n+1} := a_{n+2} := 0$.)

Proof. We apply the previous theorem for $k := (n+d, n-1, \dots, j+1, j-1, \dots, 0)$, where $d \in \{0,...,m-1\}$ and $j \in \{0,\dots,n-1\}$. Then we get that $\ell_1 = d+1$, and $\ell_i = 0$ for $i \in \{2,\dots,n\}$. Therefore,

$$\begin{aligned}
W_f^{(n+d,n-1,\dots,j+1,j-1,\dots,0)} &= W_f \Bigl| B_{d+1}\bigl(X_a,\dots,X_a^{(d)}\bigr)e_1 \quad \mathbb{I}_n e_1\\
&\qquad \dots \quad \mathbb{I}_n e_{n-j-1} \quad \mathbb{I}_n e_{n-j+1} \quad \dots \quad \mathbb{I}_n e_n \Bigr|\\
&= (-1)^{n-j-1} W_f \langle B_{d+1}\bigl(X_a,\dots,X_a^{(d)}\bigr)e_1, e_{n-j}\rangle.
\end{aligned}$$

Thus, equality (7) has been shown. In the case $d = 0$, we have that

$$\langle B_1(X_a)e_1, e_{n-j}\rangle = \langle X_a e_1, e_{n-j}\rangle = a_{n-j}$$

because the $(n-j)$th entry of X_a equals a_{n-j}. This implies the first equality in (8) for $j \in \{0,\dots,n-1\}$. In particular, for $j = n-1$, this equality is equivalent to the Abel–Liouville identity (2).

In the case $d = 1$, a simple computation gives that

$$\langle B_2(X_a, X'_a)e_1, e_{n-j}\rangle = \langle (X_a^2 + X'_a)e_1, e_{n-j}\rangle = a_1 a_{n-j} + a_{n-j+1} + a'_{n-j},$$

which yields the second equality in (8) for $j \in \{0,\dots,n-1\}$.

In the case $d = 2$, a somewhat more difficult computation gives that

$$\begin{aligned}\langle B_3(X_a, X_a', X_a'')e_1, e_{n-j}\rangle &= \langle (X_a^3 + 2X_aX_a' + X_a'X_a + X_a'')e_1, e_{n-j}\rangle \\ &= a_1^2 a_{n-j} + a_1 a_{n-j+1} + a_2 a_{n-j} + a_{n-j+2} \\ &\quad + a_1 a_{n-j}' + 2a_1' a_{n-j} + 2a_{n-j+1}' + a_{n-j}'',\end{aligned}$$

which then yields the third equality in (8). □

For the sake of convenience and brevity, we introduce the following notation: for an n-times continuously differentiable function $f\colon I \to \mathbb{R}^n$ such that W_f is nonvanishing and $j \in \{0, \dots, n-1\}$, the function $\Phi_f^{[j]}\colon I \to \mathbb{R}$ is defined by

$$\Phi_f^{[j]} := (-1)^{n-j-1} \frac{W_f^{(n,\dots,j+1,j-1,\dots,0)}}{W_f}.$$

For instance, if f is n-times continuously differentiable function whose components form a fundamental system of solutions for (1), then the Abel–Liouville identity (2) can be rewritten as

$$\Phi_f^{[n-1]} = a_1.$$

More generally, the first equality in (8) gives that

$$\Phi_f^{[j]} = a_{n-j} \quad (j \in \{0, \dots, n-1\})$$

or, equivalently,

$$a_j = \Phi_f^{[n-j]} \quad (j \in \{1, \dots, n\}). \tag{9}$$

Lemma 6. *Let $f\colon I \to \mathbb{R}^n$ be an n-times continuously differentiable function such that W_f is nonvanishing. Then the components of f form a fundamental system of solutions of the nth-order homogeneous linear differential equation*

$$y^{(n)} = \sum_{j=0}^{n-1} \Phi_f^{[j]} y^{(j)}. \tag{10}$$

Proof. This equation is equivalent to the following identity

$$\begin{vmatrix} f^{(n-1)} & \dots & f^{(0)} \end{vmatrix} y^{(n)} = \sum_{j=0}^{n-1} (-1)^{n-j-1} \begin{vmatrix} f^{(n)} & \dots & f^{(j+1)} & f^{(j-1)} & \dots & f^{(0)} \end{vmatrix} y^{(j)}.$$

We can now rearrange this equation to obtain

$$\begin{vmatrix} y^{(n)} & y^{(n-1)} & \dots & y \\ f_1^{(n)} & f_1^{(n-1)} & \dots & f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(n)} & f_n^{(n-1)} & \dots & f_n \end{vmatrix} = 0.$$

It is easily seen that if $y \in \{f_1, \dots, f_n\}$, then the determinant vanishes. Therefore, $f_1, \dots, f_n$ are solutions of (10). Due to the condition that W_f is nonvanishing, the components of f are linearly independent, therefore they form a fundamental solution system for (10). □

Corollary 7. *Let $n, m \in \mathbb{N}$ with $m \geq n$ and let $f : I \to \mathbb{R}^n$ be an m-times continuously differentiable function such that W_f is nonvanishing. Define $a = (a_1, \dots, a_n) : I \to \mathbb{R}^n$ by (9) and $X_a : I \to \mathbb{R}^{n \times n}$ by (5). Then the equality (6) holds for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, if $k_i \leq m$ and $\ell_i := (k_i - n + 1)^+$ for $i \in \{1, \dots, n\}$.*

Proof. It follows from the definition of a, that it is $(m-n)$-times continuously differentiable. On the other hand, by Lemma 6, we have that f satisfies the n-th order homogeneous linear differential equation (1). Thus, the statement is a consequence of Theorem 4. □

We say that two continuous functions $f, g : I \to \mathbb{R}^n$ are *range equivalent*, denoted by $f \sim g$, if there exists a nonsingular $n \times n$-matrix A such that

$$f = Ag. \tag{11}$$

Theorem 8. *Let $f, g : I \to \mathbb{R}^n$ be n-times continuously differentiable functions such that W_f and W_g are nonvanishing. Then $f \sim g$ holds if and only if*

$$\Phi_f^{[j]} = \Phi_g^{[j]} \quad (j \in \{0, \dots, n-1\}). \tag{12}$$

Proof. If $f \sim g$, then there exists a nonsingular $n\times n$-matrix A such that $f = Ag$. The product rule for determinants shows that $W_f^k = |A| W_g^k$ for every $k \in \mathbb{N}_0^n$. Using this identity and the definition of $\Phi_f^{[j]}$ and $\Phi_g^{[j]}$, we obtain the equalities in (12).

On the other hand, if the identities (12) are valid on I, then the nth-order homogeneous linear differential equation (10) is equivalent to the following one

$$y^{(n)} = \sum_{j=0}^{n-1} \Phi_g^{[j]} y^{(j)}.$$

Therefore, the (n-dimensional) solution spaces of these differential equations are identical, which in view of Lemma 6 yields that the components of f are linear combinations of the components of g. Thus identity (11) holds for some nonsingular $n \times n$-matrix A. □

References

[1] E.T. Bell, *Partition polynomials*, Ann. of Math. (2) **29** (1927/28), no. 1–4, 38–46.
[2] E.T. Bell, *Exponential polynomials*, Ann. of Math. (2) **35** (1934), no. 2, 258–277.
[3] R. Schimming and S.Z. Rida, *Noncommutative Bell polynomials*, Internat. J. Algebra Comput. **6** (1996), no. 5, 635–644.
[4] G. Teschl, *Ordinary Differential Equations and Dynamical Systems*, Graduate Studies in Mathematics, 140, American Mathematical Society, Providence, RI, 2012.

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