

A FUNCTIONAL EQUATION WITH BIADDITIVE FUNCTIONS

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Abstract. Let S, H, X be groups. For two given biadditive functions $A: S^2 \rightarrow X$, $B: H^2 \rightarrow X$ and for two unknown mappings $T: S \rightarrow H$, $g: S \rightarrow S$ we will study the functional equation

$$B(T(x), T(y)) = A(x, g(y)), \quad x, y \in S,$$

which is a generalization of the orthogonality equation in Hilbert spaces.

1. Introduction

Let H, K be unitary spaces. It is easy to check that, if $f: H \rightarrow K$ satisfies the orthogonality equation

$$(1.1) \quad \langle f(x)|f(y) \rangle = \langle x|y \rangle,$$

then f is a linear isometry (see, e.g. [6, Lemma 2.1.1 and the following Remark]).

The above equation was generalized in normed spaces X, Y by considering a norm derivative $\rho'_+(x, y) := \|x\| \cdot \lim_{t \rightarrow 0^+} \frac{\|x+ty\| - \|x\|}{t}$ instead of inner

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product, i.e.

$$(1.2) \quad \rho'_+(f(x), f(y)) = \rho'_+(x, y), \quad x, y \in X,$$

with an unknown function $f: X \rightarrow Y$. Note that if the norm comes from an inner product $\langle \cdot, \cdot \rangle$, we obtain $\rho'_+(x, y) = \langle x|y \rangle$.

The second way of generalization of the orthogonality equation in Hilbert spaces H, K is to look for the solutions of

$$(1.3) \quad \langle f(x)|g(y) \rangle = \langle x|y \rangle, \quad x, y \in H,$$

where $f, g: H \rightarrow K$ are unknown functions. Solutions of (1.2) and (1.3), can be found in the papers [1], [4], [2], [8].

In [5] authors give a natural generalization of such functional equations in the case of commutative groups. They consider biadditive mappings instead of inner products.

Another generalization of (1.3) we can find in the paper [7] where the author studies the equation

$$\langle f(x)|g(y^*) \rangle = \langle x|y^* \rangle, \quad x \in E, y^* \in E^*,$$

where $f: E \rightarrow F$, $g: E^* \rightarrow F^*$, E, F are Banach spaces, E^*, F^* are spaces dual to E and F respectively, and $\langle a|\varphi \rangle := \varphi(a)$.

In [3] we can find a different approach. Instead of taking two different functions on the left side of (1.3), we change only the right side of (1.3), so we obtain

$$\langle f(x)|f(y) \rangle = \langle x|g(y) \rangle, \quad x, y \in X,$$

with two unknown functions $f: X \rightarrow Y$, $g: X \rightarrow X$.

In this paper we generalize the above equation – we consider biadditive mappings instead of inner products.

2. Preliminaries

We start by recalling definition of multi-additive functions. By $\text{Perm}(n)$ we denote the set of all bijections of the set $\{1, \dots, n\}$.

DEFINITION 1. Let S be a semigroup, H be a group, $n \in \mathbb{N}$. The function $A: S^n \rightarrow H$ is called n -additive if

$$\begin{aligned} A(x_1, \dots, x_{i-1}, x_i + y, x_{i+1}, \dots, x_n) \\ = A(x_1, \dots, x_n) + A(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n), \end{aligned}$$

for all $y, x_1, \dots, x_n \in S$ and $i \in \{1, \dots, n\}$.

Moreover, A is called *symmetric* if

$$A(x_1, \dots, x_n) = A(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all $x_1, \dots, x_n \in S$ and $\sigma \in \text{Perm}(n)$.

Now we introduce some theory of the adjoint operator on groups.

DEFINITION 2. Let S, H, X be groups, $A: S^2 \rightarrow X$, $B: H^2 \rightarrow X$ be biadditive functions. Let further $T: S \rightarrow H$ and

$$D(T^*) := \{v \in H : \exists_{y \in S} \forall_{x \in S} B(T(x), v) = A(x, y)\}.$$

A function $T^*: D(T^*) \rightarrow S$ is called a (B, A) -adjoint operator (to T) if and only if

$$B(T(x), v) = A(x, T^*(v)), \quad x \in S, \quad v \in D(T^*).$$

REMARK 1. Let S, X be groups, $A: S^2 \rightarrow X$ be a biadditive function. We observe that

$$\begin{aligned} A(x, v) + A(x, y) + A(u, v) + A(u, y) &= A(x, v + y) + A(u, v + y) \\ &= A(x + u, v + y) = A(x + u, v) + A(x + u, y) \\ &= A(x, v) + A(u, v) + A(x, y) + A(u, y), \quad x, y, u, v \in S, \end{aligned}$$

so

$$A(x, y) + A(u, v) = A(u, v) + A(x, y), \quad x, y, u, v \in S.$$

Hence the group generated by the image of A is a commutative subgroup of X (so we can assume that X is commutative).

LEMMA 1 (see [5, Lemma 4]). *Let S, H, X be groups, $A: S^2 \rightarrow X, B: H^2 \rightarrow X$ be biadditive functions. Let further $T: S \rightarrow H$ and $T^*: D(T^*) \rightarrow S$ be a (B, A) -adjoint operator to T ,*

$$(2.1) \quad S_{AR} := \{y \in S : \forall_{x \in S} A(x, y) = 0\},$$

$$(2.2) \quad S_{ALT^*} := \{x \in S : \forall_{y \in \text{im } T^*} A(x, y) = 0\},$$

$$(2.3) \quad H_{BTR} := \{v \in H : \forall_{u \in \text{im } T} B(u, v) = 0\},$$

$$(2.4) \quad H_{BLD^*} := \{u \in H : \forall_{v \in D(T^*)} B(u, v) = 0\}.$$

Then

- (a) S_{AR}, S_{ALT^*} are normal subgroups of S , $D(T^*), H_{BTR}, H_{BLD^*}$ are normal subgroups of H . Moreover in the case when X is torsion-free, if H is divisible, then H_{BTR}, H_{BLD^*} are divisible, if S is divisible, then S_{AR}, S_{ALT^*} are divisible, if S, H are divisible, then $D(T^*)$ is divisible;
- (b) $\forall_{x, y \in S} T(x + y) - T(y) - T(x) \in H_{BLD^*}$;
- (c) $\forall_{x, y \in S} x - y \in S_{ALT^*} \Leftrightarrow T(x) - T(y) \in H_{BLD^*}$;
- (d) $\forall_{u, v \in D(T^*)} T^*(u + v) - T^*(v) - T^*(u) \in S_{AR}$;
- (e) $\forall_{u, v \in D(T^*)} u - v \in H_{BTR} \Leftrightarrow T^*(u) - T^*(v) \in S_{AR}$;
- (f) $H_{BTR} \subset D(T^*)$;
- (g) Let $\varkappa: S \rightarrow S/S_{AR}$ be a canonical homomorphism. Then the function $\widetilde{T^*}: D(T^*)/H_{BTR} \rightarrow \text{im } T^*/S_{AR}$ given by

$$\widetilde{T^*}(u + H_{BTR}) = T^*(u) + S_{AR}, \quad u \in D(T^*),$$

is well-defined and it is an isomorphism.

PROOF. Proofs of (b)–(f) are in [5].

Let $v \in D(T^*), x \in H$. From Remark 1 we have

$$\begin{aligned} B(T(x), x + v - x) &= B(T(x), x) + B(T(x), v) - B(T(x), x) \\ &= B(T(x), v) = B(x, T^*(v)), \end{aligned}$$

so $D(T^*)$ is a normal subgroup of G (the rest of proofs of (a) are in [5]).

Let $u, v \in D(T^*)$ be such that $u - v \in H_{BTR}$. Then from (e) we have $T^*(u) - T^*(v) \in S_{AR}$, so $\widetilde{T^*}$ is well-defined. From (d) we obtain that $\widetilde{T^*}$ is a homomorphism. From (e) we obtain the injectiveness. Obviously $\widetilde{T^*}$ is surjective. \square

REMARK 2. If in the previous lemma S, H, X are linear spaces over some field \mathbb{K} and A, B are \mathbb{K} -linear, then $S_{AR}, S_{ALT^*}, D(T^*), H_{BTR}, H_{BLD^*}$ are linear spaces over \mathbb{K} .

DEFINITION 3. Let S, H, X be groups, $A: S^2 \rightarrow X, B: H^2 \rightarrow X$ be biadditive functions, $T: S \rightarrow H$. A function T is called an (A, B) -quasi isometry if

$$(2.5) \quad B(T(x), T(y)) = A(x, y), \quad x, y \in S.$$

A function T is called an (A, B) -isometry if T is bijective and satisfies (2.5).

LEMMA 2. Let H, X be groups, \tilde{H} be a normal subgroup of $H, B: H^2 \rightarrow X$ be a biadditive function, the sets $\tilde{H}_{BL}, \tilde{H}_{BR}$ be given by formulas

$$(2.6) \quad \tilde{H}_{BL} = \{x \in \tilde{H} : \forall_{y \in \tilde{H}} B(x, y) = 0\},$$

$$(2.7) \quad \tilde{H}_{BR} = \{y \in \tilde{H} : \forall_{x \in \tilde{H}} B(x, y) = 0\},$$

$$(2.8) \quad \tilde{H}_{B0} := \tilde{H}_{BL} \cap \tilde{H}_{BR}.$$

Then $\tilde{H}_{BL}, \tilde{H}_{BR}$, and \tilde{H}_{B0} are normal subgroups of \tilde{H} , the function $\tilde{B}: (\tilde{H}/\tilde{H}_{B0})^2 \rightarrow X$ given by the formula

$$(2.9) \quad \tilde{B}(x + \tilde{H}_{B0}, y + \tilde{H}_{B0}) := B(x, y), \quad x, y \in \tilde{H},$$

is well-defined and it is biadditive.

PROOF. It is easy to observe that $\tilde{H}_{BL}, \tilde{H}_{BR}, \tilde{H}_{B0}$ are normal subgroups of \tilde{H} . Let $x_1, x_2, y_1, y_2 \in \tilde{H}$ be such that $x_2 - x_1, y_2 - y_1 \in \tilde{H}_{B0}$. Then

$$\begin{aligned} B(x_2, y_2) &= B(x_2 - x_1, y_2) + B(x_1, y_2) = B(x_1, y_2) \\ &= B(x_1, y_2 - y_1) + B(x_1, y_1) = B(x_1, y_1), \end{aligned}$$

so \tilde{B} is well-defined. It is easy to observe that \tilde{B} is biadditive. □

Of course each (A, B) -isometry is an (A, B) -quasi isometry. The following result shows that for any (A, B) -quasi isometry there exists some (\tilde{A}, \tilde{B}) -isometry connected with it.

THEOREM 1. Let S, H, X are groups, $A: S^2 \rightarrow X$, $B: H^2 \rightarrow X$ be biadditive functions, $T: S \rightarrow H$ be an (A, B) -quasi isometry, $\tilde{H} = \langle \text{im } T \rangle$. Let further \tilde{H}_{B0} , S_{AR} be defined by (2.8) and (2.1),

$$S_{AL} := \{x \in S : \forall_{y \in S} A(x, y) = 0\},$$

$$S_{A0} = S_{AL} \cap S_{AR}.$$

Then S_{AL} , S_{A0} are normal subgroups of S , the function $\tilde{T}: S/S_{A0} \rightarrow \tilde{H}/\tilde{H}_{B0}$ given by the formula

$$\tilde{T}(x + S_{A0}) := T(x) + \tilde{H}_{B0}, \quad x \in S,$$

is well-defined and it is an (\tilde{A}, \tilde{B}) -isometry, where \tilde{A} , \tilde{B} are defined by (2.9) (for \tilde{A} we take $B = A$, $\tilde{H} = S$ in the previous lemma).

PROOF. It is easy to observe that S_{AL} , S_{A0} are normal subgroups of S . Let $x, y \in S$. We observe that

$$\begin{aligned} A(x - y, z) &= A(x, z) - A(y, z) = B(T(x), T(z)) - B(T(y), T(z)) \\ &= B(T(x) - T(y), T(z)), \end{aligned}$$

so $x - y \in S_{AL} \Leftrightarrow T(x) - T(y) \in \tilde{H}_{BL}$. In analogical way we can obtain that $x - y \in S_{AR} \Leftrightarrow T(x) - T(y) \in \tilde{H}_{BR}$. Hence $x - y \in S_{A0} \Leftrightarrow T(x) - T(y) \in \tilde{H}_{B0}$, so \tilde{T} is well-defined and it is injective.

Let $v \in \tilde{H}$, then there exist $k_1, \dots, k_n \in \mathbb{Z}$, $x_1, \dots, x_n \in S$ such that $v = \sum_{i=1}^n k_i T(x_i)$. Since for $x, y, z \in S$ we have

$$\begin{aligned} 0 &= A(x + y - y - x, z) = A(x + y, z) - A(y, z) - A(x, z) \\ &= B(T(x + y), T(z)) - B(T(y), T(z)) - B(T(x), T(z)) \\ &= B(T(x + y) - T(y) - T(x), T(z)), \\ 0 &= A(z, x + y - y - x) = A(z, x + y) - A(z, y) - A(z, x) \\ &= B(T(z), T(x + y)) - B(T(z), T(y)) - B(T(z), T(x)) \\ &= B(T(z), T(x + y) - T(y) - T(x)), \end{aligned}$$

then $T(x+y) - T(y) - T(x) \in \tilde{H}_{B0}$. Hence we have

$$v + \tilde{H}_{B0} = \sum_{i=1}^n k_i T(x_i) + \tilde{H}_{B0} = T\left(\sum_{i=1}^n k_i x_i\right) + \tilde{H}_{B0} = \tilde{T}\left(\sum_{i=1}^n k_i x_i + S_{A0}\right),$$

which means that \tilde{T} is surjective. \square

3. Main results

In this section we assume that S, H, X are groups, $A: S^2 \rightarrow X$, $B: H^2 \rightarrow X$ are biadditive functions, \tilde{H} is a normal subgroup of H , \tilde{H}_{BL} , \tilde{H}_{BR} , \tilde{H}_{B0} , S_{AR} are given resp. by (2.6), (2.7), (2.8), (2.1).

REMARK 3. Let $T: S \rightarrow H$, $g: S \rightarrow S$ satisfy the equation

$$(3.1) \quad B(T(x), T(y)) = A(x, g(y)), \quad x, y \in S.$$

Then we can assume that H is generated by $\text{im } T$.

THEOREM 2. Let $T: S \rightarrow H$, $g: S \rightarrow S$ satisfy equation (3.1), $\tilde{H} := \langle \text{im } T \rangle$, $T^*: H \rightarrow S$ be a (B, A) -adjoint operator to T . Then $\tilde{H} \subset D(T^*)$,

$$T^*(T(y)) - g(y) \in S_{AR}, \quad y \in S.$$

Moreover, if $H = \tilde{H}$ then $H_{BLD^*} = \tilde{H}_{BL}$, $H_{BTR} = \tilde{H}_{BR}$, where H_{BTR} , H_{BLD^*} are given respectively by (2.3), (2.4).

PROOF. From (3.1) we obtain that $\text{im } T \subset D(T^*)$, so $\tilde{H} \subset D(T^*)$ and

$$\begin{aligned} A(x, T^*(T(y)) - g(y)) &= A(x, T^*(T(y))) - A(x, g(y)) \\ &= B(T(x), T(y)) - B(T(x), T(y)) = 0, \quad x, y \in S. \end{aligned}$$

Hence $T^*(T(y)) - g(y) \in S_{AR}$.

Assume that $H = \tilde{H}$. We notice that $H_{BR} \subset H_{BTR}$ and since $H = D(T^*)$, we get $H_{BL} = H_{BLD^*}$. Let $v \in H_{BTR}$. For $u \in H$ there exist $x_1, \dots, x_n \in S$ and $k_1, \dots, k_n \in \mathbb{Z}$ such that $u = \sum_{i=1}^n k_i T(x_i)$. Then

$$B(u, v) = B\left(\sum_{i=1}^n k_i T(x_i), v\right) = \sum_{i=1}^n k_i B(T(x_i), v) = 0,$$

so $v \in H_{BR}$. □

THEOREM 3. *Let $T: S \rightarrow H$, $g: S \rightarrow S$ satisfy equation (3.1), $\tilde{H} := \langle \text{im } T \rangle$, $T^*: H \rightarrow S$ be a (B, A) -adjoint operator to T , $\tilde{H}_{BL} \subset \tilde{H}_{BR}$. Then*

$$(3.2) \quad S_{ALT^*} \subset S_{ALg} := \{x \in S : \forall_{y \in S} A(x, g(y)) = 0\},$$

$$(3.3) \quad g(x + y) - g(y) - g(x) \in S_{AR}, \quad x, y \in S,$$

$$g(S_{ALT^*}) \subset S_{AR},$$

and T is an (A_1, B) -quasi isometry, where S_{ALT^*} is given by (2.2) and $A_1: S^2 \rightarrow X$ is given by the formula

$$(3.4) \quad A_1(x, y) = A(x, g(y)), \quad x, y \in S.$$

Moreover, if $\tilde{H} = H$, then $S_{ALT^*} = S_{ALg}$.

PROOF. Using previous theorem we get $\langle \text{im } g \rangle + S_{AR} \subset \langle \text{im } T^* \rangle + S_{AR}$, so $S_{ALT^*} \subset S_{ALg}$ (when $\tilde{H} = H$ we have $\langle \text{im } g \rangle + S_{AR} = \langle \text{im } T^* \rangle + S_{AR}$, so $S_{ALT^*} = S_{ALg}$).

Using Lemma 1 (b) we have also

$$\begin{aligned} A(z, g(x + y) - g(y) - g(x)) &= A(z, g(x + y)) - A(z, g(y)) - A(z, g(x)) \\ &= B(T(z), T(x + y)) - B(T(z), T(y)) - B(T(z), T(x)) \\ &= B(T(z), T(x + y) - T(y) - T(x)) = 0, \quad x, y, z \in S. \end{aligned}$$

Hence $g(x + y) - g(y) - g(x) \in S_{AR}$ for $x, y \in S$.

Let $x \in S$, $y \in S_{ALT^*}$. Then in view of Lemma 1 (c) we get $T(y) \in H_{BLD^*}$, so $T(y) \in \tilde{H}_{BL} \subset \tilde{H}_{BR}$. Hence we have

$$A(x, g(y)) = B(T(x), T(y)) = 0,$$

which means that $g(y) \in S_{AR}$.

For $x, y \in S$ we have

$$B(T(x), T(y)) = A(x, g(y)) = A_1(x, y),$$

which ends the proof. \square

The following example shows that the assumption $\tilde{H}_{BL} \subset \tilde{H}_{BR}$ is important in the previous theorem.

EXAMPLE 1. Let $S = H = \mathbb{Q}^2$, $X = \mathbb{Q}$, $g = (g_1, g_2): S \rightarrow S$ be an arbitrary function, $f: S \rightarrow H$ be a function given by the formula

$$f(x) = (x_1 + x_2, g_2(x)), \quad x = (x_1, x_2) \in \mathbb{Q}^2.$$

Let further $B: H^2 \rightarrow X$, $A: S^2 \rightarrow X$ be functions given by formulas

$$B((x_1, x_2), (y_1, y_2)) = x_1 y_2, \quad x_1, x_2, y_1, y_2 \in \mathbb{Q},$$

$$A((x_1, x_2), (y_1, y_2)) = (x_1 + x_2) y_2, \quad x_1, x_2, y_1, y_2 \in \mathbb{Q}.$$

It is easy to see that A, B are biadditive and $S_{AR} = \mathbb{Q} \times 0$.

We have also

$$\begin{aligned} A(x, g(y)) &= (x_1 + x_2) g_2(y) = B((x_1 + x_2, g_2(x)), (y_1 + y_2, g_2(y))) \\ &= B(f(x), f(y)), \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{Q}^2. \end{aligned}$$

Hence (3.1) holds. Of course g_2 can be nonadditive, so there exist $x, y \in S$ such that $g(x + y) - g(y) - g(x) \notin S_{AR}$.

It is a natural question whether given a function T there exists a function g such that (T, g) satisfies equation (3.1). The lemma below gives us an answer for this question.

LEMMA 3. Let $T: S \rightarrow H$ be such that $\text{im } T \subset D(T^*)$, where T^* is a (B, A) -adjoint operator to T . Then (T, g) satisfies equation (3.1), where $g = T^* \circ T: S \rightarrow S$.

PROOF. We observe that

$$B(T(x), T(y)) = A(x, T^*(T(y))) = A(x, g(y)), \quad x, y \in S,$$

which ends the proof. \square

The second natural question is whether given a function g there exists a function T such that (T, g) satisfies equation (3.1).

LEMMA 4. *Let $g: S \rightarrow S$ satisfies conditions (3.3), $g(S_{ALg}) \subset S_{AR}$ and there exists a subgroup \tilde{H} of H such that there exists an (A_1, B) -quasi isometry $T: S \rightarrow \tilde{H}$ where A_1, S_{ALg} are given resp. by (3.4), (3.2). Then (T, g) satisfies (3.1).*

PROOF. Let $x, y \in S$. Then

$$B(T(x), T(y)) = A_1(x, y) = A(x, g(y)),$$

which ends the proof. \square

In the previous lemma, in the case when S, H are unitary spaces and A, B are inner products, instead assuming the existence of an (A_1, B) -quasi isometry we can assume that g is positive and symmetric and there exists an isometry from S to some subspace of H (see [3, Theorem 8]). In general we do not have a similar result but we can write here the following

THEOREM 4. *Let $g, h: S \rightarrow S$ be such that $\text{im } h \subset D(h^*)$ and $g(x) - (h^* \circ h)(x) \in S_{AR}$ for $x \in S$, where $h^*: D(h^*) \rightarrow S$ is an (A, A) -adjoint operator to h . Assume that $\text{im } h$ is (B, A) -isometric with some subset of H , i.e. there exists $I: S \rightarrow H$ such that*

$$B(I(h(x)), I(h(y))) = A(h(x), h(y)), \quad x, y \in S.$$

Then (T, g) satisfies (3.1) with $T := I \circ h$.

PROOF. We observe that

$$\begin{aligned} B(T(x), T(y)) &= B(I(h(x)), I(h(y))) = A(h(x), h(y)) \\ &= A(x, (h^* \circ h)(y)) = A(x, g(y)), \quad x, y \in S. \end{aligned} \quad \square$$

The following example shows that we cannot reverse the previous theorem.

EXAMPLE 2. Let $S = H = X = \mathbb{Q}$, $A: S^2 \rightarrow X$, $B: H^2 \rightarrow X$, $g: S \rightarrow S$, $T: S \rightarrow H$ be maps given by formulas

$$\begin{aligned} A(x, y) &= xy, \quad x, y \in S, \quad B(x, y) = -xy, \quad x, y \in S, \\ g(x) &= -x, \quad x \in S, \quad T(x) = x, \quad x \in S. \end{aligned}$$

Then it is easy to see that (T, g) satisfies (3.1). Suppose that there exists a map $h: S \rightarrow S$ such that $\text{im } h \subset D(h^*)$, $g(x) - h^* \circ h(x) \in S_{AR}$, where $h^*: D(h^*) \rightarrow S$ is (A, A) -adjoint operator to h . Then

$$\begin{aligned} 0 \leq h(x)^2 &= A(h(x), h(x)) = A(x, (h^* \circ h)(x)) \\ &= A(x, g(x)) = A(x, -x) = -x^2, \quad x \in S, \end{aligned}$$

which gives us a contradiction.

We can also say something about the family of functions T which satisfy equation (3.1) with the same g .

THEOREM 5. *Let $T_1, T: S \rightarrow H$, $g: S \rightarrow S$, (T, g) satisfies (3.1), $\tilde{H} := \langle \text{im } T_1 \rangle$. Then (T_1, g) satisfies (3.1) if and only if there exists a $(B|_{\langle \text{im } T \rangle^2}, B)$ -quasi isometry $I: \langle \text{im } T \rangle \rightarrow \langle \text{im } T_1 \rangle$ such that $T_1(x) - I(T(x)) \in \tilde{H}_{B0}$ for $x \in S$.*

PROOF. Assume that (T_1, g) satisfies (3.1). For $x \in \langle \text{im } T \rangle$ let $\varphi(x)$ be an arbitrary element of the set $x + \tilde{H}_{B0}$. We define $I: \langle \text{im } T \rangle \rightarrow \tilde{H}$ by the formula

$$I\left(\sum_{i=1}^n k_i T(x_i)\right) := \varphi\left(\sum_{i=1}^n k_i T_1(x_i) + \tilde{H}_{B0}\right), \quad k_1, \dots, k_n \in \mathbb{Z}, \quad x_1, \dots, x_n \in S.$$

Let $k_1, \dots, k_n, r_1, \dots, r_n \in \mathbb{Z}$, $y, x_1, \dots, x_n \in S$ and $\sum_{i=1}^n k_i T(x_i) = \sum_{i=1}^n r_i T(x_i)$. Then

$$\begin{aligned} 0 &= B\left(\sum_{i=1}^n k_i T(x_i) - \sum_{i=1}^n r_i T(x_i), T(y)\right) \\ &= \sum_{i=1}^n k_i B(T(x_i), T(y)) - \sum_{i=1}^n r_i B(T(x_i), T(y)) \\ &= \sum_{i=1}^n k_i A(x_i, g(y)) - \sum_{i=1}^n r_i A(x_i, g(y)) \\ &= \sum_{i=1}^n k_i B(T_1(x_i), T_1(y)) - \sum_{i=1}^n r_i B(T_1(x_i), T_1(y)) \\ &= B\left(\sum_{i=1}^n k_i T_1(x_i) - \sum_{i=1}^n r_i T_1(x_i), T_1(y)\right), \end{aligned}$$

so $\sum_{i=1}^n k_i T_1(x_i) - \sum_{i=1}^n r_i T_1(x_i) \in \widetilde{H}_{BL}$. In analogical way we can prove that

$$\sum_{i=1}^n k_i T_1(x_i) - \sum_{i=1}^n r_i T_1(x_i) \in \widetilde{H}_{BR}.$$

Hence $\sum_{i=1}^n k_i T_1(x_i) - \sum_{i=1}^n r_i T_1(x_i) \in \widetilde{H}_0$, so I is well-defined. We have also

$$\begin{aligned} B\left(\sum_{i=1}^n k_i T(x_i), \sum_{j=1}^n r_j T(x_j)\right) &= \sum_{i=1}^n k_i \sum_{j=1}^n r_j B(T(x_i), T(x_j)) \\ &= \sum_{i=1}^n k_i \sum_{j=1}^n r_j B(T_1(x_i), T_1(x_j)) \\ &= B\left(\sum_{i=1}^n k_i T_1(x_i), \sum_{j=1}^n r_j T_1(x_j)\right) \\ &= B\left(\varphi\left(\sum_{i=1}^n k_i T_1(x_i)\right), \varphi\left(\sum_{j=1}^n r_j T_1(x_j)\right)\right), \end{aligned}$$

so I is a $(B|_{\langle \text{im } T \rangle^2}, B)$ -quasi isometry. We have also

$$I(T(x)) + \widetilde{H}_{B0} = \varphi(T_1(x) + \widetilde{H}_{B0}) + \widetilde{H}_{B0} = T_1(x) + \widetilde{H}_{B0}, \quad x \in S,$$

so $T_1(x) - I(T(x)) \in \widetilde{H}_{B0}$ for $x \in S$.

Let $I: \langle \text{im } T \rangle \rightarrow \widetilde{H}$ be a $(B|_{\langle \text{im } T \rangle^2}, B)$ -quasi isometry such that $T_1(x) - I(T(x)) \in \widetilde{H}_{B0}$ for $x \in S$. Then

$$\begin{aligned} B(T_1(x), T_1(y)) &= B(I(T(x)), I(T(y))) \\ &= B(T(x), T(y)) = A(x, g(y)), \quad x, y \in S, \end{aligned}$$

which ends the proof. □

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