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ON GENERALIZED JACOBSTHAL AND JACOBSTHAL–LUCAS NUMBERS

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Abstract. Jacobsthal numbers and Jacobsthal–Lucas numbers are some of the most studied special integer sequences related to the Fibonacci numbers. In this study, we introduce one parameter generalizations of Jacobsthal numbers and Jacobsthal–Lucas numbers. We define two sequences, called generalized Jacobsthal sequence and generalized Jacobsthal–Lucas sequence. We give generating functions, Binet's formulas for these numbers. Moreover, we obtain some identities, among others Catalan's, Cassini's identities and summation formulas for the generalized Jacobsthal numbers and the generalized Jacobsthal–Lucas numbers. These properties generalize the well-known results for classical Jacobsthal numbers and Jacobsthal–Lucas numbers. Additionally, we give a matrix representation of the presented numbers.

1. Introduction

The Jacobsthal sequence $\{J_n\}$ is defined recursively in the following way

(1.1) $$J_0 = 0,\ J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2.$$

The Jacobsthal–Lucas sequence $\{j_n\}$ is defined by the same recurrence

(1.2) $$j_n = j_{n-1} + 2j_{n-2} \quad \text{for } n \geq 2$$

with $j_0 = 2$, $j_1 = 1$.

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The Binet's formulas of these sequences have the following form

$$J_n = \frac{1}{3}(2^n - (-1)^n), \quad j_n = 2^n + (-1)^n.$$

Some interesting properties of the Jacobsthal numbers are given in [6]. There are many generalizations of this sequence in the literature. We recall some of such generalizations:

1) k-Jacobsthal sequence $\{j_{k,n}\}$ ([7]), $j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $j_{k,0} = 0$, $j_{k,1} = 1$,
2) k-Jacobsthal sequence $\{J_{k,n}\}$ ([4]), $J_{k,n+1} = J_{k,n} + kJ_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $J_{k,0} = 0$, $J_{k,1} = 1$,
3) Jacobsthal r-sequence $\{J(r,n)\}$ ([2]), for $r \geq 0$ $J(r,n) = 2^r J(r,n-1) + (2^r + 4^r)J(r,n-2)$ for $n \geq 2$ with $J(r,0) = 1$, $J(r,1) = 1 + 2^{r+1}$.
4) Jacobsthal (s,p)-sequence $\{J_n(s,p)\}$ ([1]), for $s,p \geq 0$, $n \geq 2$ $J_n(s,p) = 2^{s+p}J_{n-1}(s,p) + (2^{2s+p} + 2^{s+2p})J_{n-2}(s,p)$ with $J_0(s,p) = 1$, $J_1(s,p) = 2^s + 2^p + 2^{s+p}$.
5) Jacobsthal sequence $\{J(d,t,n)\}$ ([10]), $J(d,t,n) = J(d,t,n-1) + tJ(d,t,n-d)$ for $n \geq d$ with $J(d,t,0) = 1$, $J(d,t,n) = 1$ for $n = 1,\ldots,d$, $t \geq 1, d \geq 2$.

More considerations concerning certain generalizations of Jacobsthal and Jacobsthal–Lucas numbers were given, among others, in [3, 5, 8, 11].

We introduce a new generalization of classical Jacobsthal and Jacobsthal--Lucas numbers. This generalization depends on one integer parameter k used in the recurrence relation (1.1). We will show some interesting properties of these numbers. They generalize known properties of the numbers J_n and j_n.

2. Generalized Jacobsthal and Jacobsthal–Lucas numbers

Let $n \geq 0$, $k \geq 2$ be integers. Generalized Jacobsthal sequence $\{J(k,n)\}$ is defined by the recurrence

$$J(k,n) = (k-1)J(k,n-1) + kJ(k,n-2) \quad \text{for } n \geq 2 \tag{2.1}$$

with initial conditions $J(k,0) = 0$, $J(k,1) = 1$. Generalized Jacobsthal–Lucas sequence $\{j(k,n)\}$ we define using the same recurrence relation

$$j(k,n) = (k-1)j(k,n-1) + kj(k,n-2) \quad \text{for } n \geq 2 \tag{2.2}$$

with $j(k,0) = 2$, $j(k,1) = 1$.

For $k=2$ we obtain $J(2,n)=J_n$ and $j(2,n)=j_n$. By (2.1) we get

$$\begin{aligned}
&J(k,2)=k-1\\
&J(k,3)=k^2-k+1\\
&J(k,4)=k^3-k^2+k-1\\
&J(k,5)=k^4-k^3+k^2-k+1\\
&J(k,6)=k^5-k^4+k^3-k^2+k-1\\
&J(k,7)=k^6-k^5+k^4-k^3+k^2-k+1\\
&\vdots
\end{aligned}$$

By (2.2) we have

$$\begin{aligned}
&j(k,2)=3k-1\\
&j(k,3)=3k^2-3k+1\\
&j(k,4)=3k^3-3k^2+3k-1\\
&j(k,5)=3k^4-3k^3+3k^2-3k+1\\
&j(k,6)=3k^5-3k^4+3k^3-3k^2+3k-1\\
&j(k,7)=3k^6-3k^5+3k^4-3k^3+3k^2-3k+1\\
&\vdots
\end{aligned}$$

By the recurrences (2.1) and (2.2) we obtain the following result.

Proposition 2.1. *Let $n\geq 4$, $k\geq 2$. Then*

(i) $J(k,n)=(k^2+1)(k-1)J(k,n-3)+k(k^2-k+1)J(k,n-4)$,

(ii) $j(k,n)=(k^2+1)(k-1)j(k,n-3)+k(k^2-k+1)j(k,n-4)$.

Proof. (i) By the formula (2.1) we obtain

$$\begin{aligned}
J(k,n)&=(k-1)J(k,n-1)+kJ(k,n-2)\\
&=(k-1)\big[(k-1)J(k,n-2)+kJ(k,n-3)\big]+kJ(k,n-2)\\
&=(k^2-k+1)\big[(k-1)J(k,n-3)+kJ(k,n-4)\big]\\
&+(k-1)kJ(k,n-3)\\
&=(k^2+1)(k-1)J(k,n-3)+k(k^2-k+1)J(k,n-4).
\end{aligned}$$

The proof of (ii) is analogous. $\square$

It is easily seen that by the recurrence relation (2.1) for $k = 2, 3, \ldots$ we get well-known sequences, see [9]. For example, sequences $\{J(2, n)\}$, $\{J(3, n)\}$, $\{J(4, n)\}$, $\{J(5, n)\}$, $\{J(6, n)\}$, $\{J(7, n)\}$ are listed under the symbols A001045, A015518, A015521, A015531, A015540, A015552, respectively.

Tables 1 and 2 include a few first terms of the sequences $\{J(k, n)\}$ and $\{j(k, n)\}$ for special k and n.

Table 1. The terms of $\{J(k, n)\}$ for $2 \le k \le 7$ and $n \le 8$

n	0	1	2	3	4	5	6	7	8
$J(2, n)$	0	1	1	3	5	11	21	43	85
$J(3, n)$	0	1	2	7	20	61	182	547	1640
$J(4, n)$	0	1	3	13	51	205	819	3277	13107
$J(5, n)$	0	1	4	21	104	521	2604	13021	65104
$J(6, n)$	0	1	5	31	185	1111	6665	3991	239945
$J(7, n)$	0	1	6	43	300	2101	14706	102943	720600

Table 2. The terms of $\{j(k, n)\}$ for $2 \le k \le 7$ and $n \le 8$

n	0	1	2	3	4	5	6	7	8
$j(2, n)$	2	1	5	7	17	31	65	127	257
$j(3, n)$	2	1	8	19	62	181	548	1639	4922
$j(4, n)$	2	1	11	37	155	613	2459	9829	39323
$j(5, n)$	2	1	14	61	314	1561	7814	39061	195314
$j(6, n)$	2	1	17	91	557	3331	19997	119971	719837
$j(7, n)$	2	1	20	127	902	6301	44120	308827	2161802

We can modify the equality (2.1). Instead of the coefficients $k-1$, k we can give new coefficients such that the obtained recurrence will generalize classical Jacobsthal numbers. For example, for $s \in \mathbb{N}_0$ let

$$\mathcal{J}(s, n) = (s+1)\mathcal{J}(s, n-1) + (2s+2)\mathcal{J}(s, n-2) \quad \text{for } n \ge 2 \tag{2.3}$$

with $\mathcal{J}(s, 0) = 0$, $\mathcal{J}(s, 1) = 1$. For $s = 0$ we obtain $\mathcal{J}(0, n) = J_n$. Table 3 includes a few first terms of the sequence $\{\mathcal{J}(s, n)\}$ for special s and n.

Table 3. The terms of $\{\mathcal{J}(s, n)\}$ for $0 \le s \le 5$ and $n \le 8$

n	0	1	2	3	4	5	6	7	8
$\mathcal{J}(0, n)$	0	1	1	3	5	11	21	43	85
$\mathcal{J}(1, n)$	0	1	2	8	24	80	256	832	2688
$\mathcal{J}(2, n)$	0	1	3	15	63	279	1215	5319	23247
$\mathcal{J}(3, n)$	0	1	4	24	128	704	3840	20992	114688
$\mathcal{J}(4, n)$	0	1	5	35	225	1475	9625	62875	410625
$\mathcal{J}(5, n)$	0	1	6	48	360	2736	20736	157248	1192320

Now we present the Binet's formulas of the numbers $J(k,n)$ and $j(k,n)$. The characteristic equation of (2.1) is

$$r^2 - (k-1)r - k = 0. \tag{2.4}$$

Since $\Delta = (k+1)^2 > 0$ for $k \geq 2$, we get two real roots of (2.4)

$$r_1 = -1, \quad r_2 = k.$$

Clearly,

$$r_1 + r_2 = k - 1, \tag{2.5}$$

$$r_2 - r_1 = k + 1, \tag{2.6}$$

$$r_1 r_2 = -k. \tag{2.7}$$

Theorem 2.1 (Binet's formula). *Let $n \geq 0$, $k \geq 2$. Then*

$$J(k,n) = \frac{r_2^n - r_1^n}{r_2 - r_1} = \frac{k^n - (-1)^n}{k+1}, \tag{2.8}$$

$$j(k,n) = \frac{(2r_2 - 1)r_1^n + (1 - 2r_1)r_2^n}{r_2 - r_1} = \frac{(2k-1)(-1)^n + 3k^n}{k+1}. \tag{2.9}$$

Proof. The nth generalized Jacobsthal number may be written in the following form

$$J(k,n) = C_1(-1)^n + C_2 k^n$$

for some constants C_1 and C_2. Using initial conditions of the recurrence (2.1), we obtain the following system of two equations

$$\begin{cases} C_1 + C_2 = 0, \\ -C_1 + kC_2 = 1. \end{cases}$$

Hence

$$C_1 = -\frac{1}{1+k} \quad \text{and} \quad C_2 = \frac{1}{1+k}.$$

Thus

$$J(k,n) = \frac{k^n - (-1)^n}{k+1} = \frac{r_2^n - r_1^n}{r_2 - r_1}.$$

In order to prove (2.9) we write

$$j(k,n) = c_1(-1)^n + c_2k^n.$$

By (2.2) we get

$$\begin{cases} c_1 + c_2 = 2, \\ -c_1 + kc_2 = 1. \end{cases}$$

Hence

$$c_1 = \frac{2k-1}{1+k} \quad \text{and} \quad c_2 = \frac{3}{1+k}$$

and the result follows. □

THEOREM 2.2. *Let $n \geq 0$, $k \geq 2$. Then*

$$3\,J(k,n) = 2(-1)^{n+1} + j(k,n).$$

PROOF. By (2.8) and (2.9) we get

$$\begin{aligned} 3J(k,n) - j(k,n) &= \frac{3(k^n - (-1)^n) - (2k-1)(-1)^n - 3k^n}{k+1} \\ &= \frac{-(-1)^n(2k+2)}{k+1} = 2(-1)^{n+1}, \end{aligned}$$

which completes the proof. □

3. Some identities for the sequences $\{J(k,n)\}$ and $\{j(k,n)\}$

Now we present interesting properties of the numbers $J(k,n)$ and $j(k,n)$.

THEOREM 3.1. *Let $k \geq 2$ be an integer. Then*

$$\lim_{n\to\infty} \frac{J(k,n+1)}{J(k,n)} = k, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{j(k,n+1)}{j(k,n)} = k. \tag{3.2}$$

Proof. Using Theorem 2.1, we have

$$\lim_{n\to\infty}\frac{J(k,n+1)}{J(k,n)}=\lim_{n\to\infty}\frac{C_1{r_1}^{n+1}+C_2r_2^{n+1}}{C_1{r_1}^{n}+C_2r_2^{n}}=\lim_{n\to\infty}\frac{C_1r_1\left(\frac{r_1}{r_2}\right)^n+C_2r_2}{C_1\left(\frac{r_1}{r_2}\right)^n+C_2}.$$

Since $\lim\limits_{n\to\infty}(\frac{r_1}{r_2})^n=0$, we get

$$\lim_{n\to\infty}\frac{J(k,n+1)}{J(k,n)}=r_2=k.$$

The proof of (3.2) is analogous. □

Theorem 3.2 (Catalan's identity). *Let $n\geq 1$, $k\geq 2$, $r\geq 0$ be integers. Then*

$$J(k,n-r)J(k,n+r)-J^2(k,n)=(-1)^{n-r+1}k^{n-r}J^2(k,r).$$

Proof. By formula (2.8) we obtain

$$\begin{aligned}
&J(k,n-r)J(k,n+r)-J^2(k,n)\\
&\qquad=\frac{(r_2^{n-r}-r_1^{n-r})\cdot(r_2^{n+r}-r_1^{n+r})}{(r_2-r_1)^2}-\left(\frac{r_2^n-r_1^n}{r_2-r_1}\right)^2\\
&\qquad=\frac{2r_1^nr_2^n-r_1^{n+r}r_2^{n-r}-r_1^{n-r}r_2^{n+r}}{(r_2-r_1)^2}.
\end{aligned}$$

After simple calculations, using formulas (2.6) and (2.7), we get

$$\begin{aligned}
&J(k,n-r)J(k,n+r)-J^2(k,n)\\
&\qquad=\frac{(r_1r_2)^n}{(r_2-r_1)^2}\left(2-\left(\frac{r_1}{r_2}\right)^r-\left(\frac{r_2}{r_1}\right)^r\right)\\
&\qquad=\frac{(-k)^n}{(r_2-r_1)^2}\left(2-\frac{r_1^{2r}+r_2^{2r}}{(r_1r_2)^r}\right)\\
&\qquad=-\frac{(-k)^n}{(-k)^r(r_2-r_1)^2}\cdot\left(r_1^{2r}-2(r_1r_2)^r+r_2^{2r}\right)\\
&\qquad=-(-k)^{n-r}\left(\frac{r_2^r-r_1^r}{r_2-r_1}\right)^2=(-1)^{n-r+1}k^{n-r}J^2(k,r).
\end{aligned}$$

The proof is complete. □

Corollary 3.1 (Cassini's identity). *Let $n \geq 1$, $k \geq 2$. Then*

$$J(k,n-1)J(k,n+1) - J^2(k,n) = (-1)^n k^{n-1}. \tag{3.3}$$

By formula (3.3), taking $k = 2$, we obtain Cassini's formula for the numbers J_n.

Corollary 3.2. *For $n \geq 1$ we have*

$$J_{n-1}J_{n+1} - J_n^2 = (-1)^n 2^{n-1}.$$

Theorem 3.3 (Catalan's identity). *Let $n \geq 1$, $k \geq 2$, $r \geq 0$. Then*

$$j(k,n-r)j(k,n+r) - j^2(k,n) = (6k-3)(-1)^{n-r}k^{n-r}J^2(k,r).$$

Proof. By (2.9) we have

$$j(k,n) = c_1 r_1^n + c_2 r_2^n, \tag{3.4}$$

where $c_1 = \frac{2k-1}{k+1}$, $c_2 = \frac{3}{k+1}$. Hence

$$\begin{aligned}
j(k,n-r)&j(k,n+r) - j^2(k,n)\\
&= (c_1 r_1^{n-r} + c_2 r_2^{n-r})(c_1 r_1^{n+r} + c_2 r_2^{n+r}) - (c_1 r_1^n + c_2 r_2^n)^2\\
&= c_1 c_2 (r_1^{n-r} r_2^{n+r} + r_1^{n+r} r_2^{n-r} - 2(r_1 r_2)^n)\\
&= c_1 c_2 (r_1 r_2)^n \left[\left(\frac{r_1}{r_2}\right)^r + \left(\frac{r_2}{r_1}\right)^r - 2\right]\\
&= \frac{(2r_2-1)(1-2r_1)}{(r_2-r_1)^2}(r_1 r_2)^n \left[\frac{r_1^{2r} - 2(r_1 r_2)^r + r_2^{2r}}{(r_1 r_2)^r}\right]\\
&= (2r_2-1)(1-2r_1)(r_1 r_2)^{n-r}\left(\frac{r_2^r - r_1^r}{r_2 - r_1}\right)^2.
\end{aligned}$$

By (2.7) and (2.8) we get

$$\begin{aligned}
j(k,n-r)j(k,n+r) - j^2(k,n) &= (2r_2-1)(1-2r_1)(-k)^{n-r}J^2(k,r)\\
&= (6k-3)(-1)^{n-r}k^{n-r}J^2(k,r). \qquad \square
\end{aligned}$$

Corollary 3.3 (Cassini's identity). *For $n \geq 1$ and $k \geq 2$ we have*

$$j(k,n-1)j(k,n+1) - j^2(k,n) = (6k-3)(-1)^{n-1}k^{n-1}. \tag{3.5}$$

By (3.5) we obtain Cassini's identity for the numbers j_n.

Corollary 3.4. *For $n \geq 1$ we have*

$$j_{n-1}j_{n+1} - j_n^2 = 9(-1)^{n-1}2^{n-1}.$$

Theorem 3.4. *Let k, m, n be integers and $n, m \geq 1$, $m \geq n$, $k \geq 2$. Then*

$$J(k,m)J(k,n+1) - J(k,m+1)J(k,n) = (-k)^n J(k,m-n).$$

Proof. By (2.8) we obtain

$$\begin{aligned}
J(k,m)&J(k,n+1) - J(k,m+1)J(k,n)\\
&= \frac{(r_2^m - r_1^m)\cdot(r_2^{n+1} - r_1^{n+1})}{(r_2-r_1)^2} - \frac{(r_2^{m+1} - r_1^{m+1})\cdot(r_2^n - r_1^n)}{(r_2-r_1)^2}\\
&= -\frac{1}{(r_2-r_1)^2}\left(r_1^{n+1}r_2^m + r_1^m r_2^{n+1} - r_1^n r_2^{m+1} - r_1^{m+1}r_2^n\right)\\
&= \frac{1}{(r_2-r_1)^2}\big(r_1^n r_2^m(r_2-r_1) - r_1^m r_2^n(r_2-r_1)\big)\\
&= (r_1r_2)^n \frac{r_2^{m-n} - r_1^{m-n}}{r_2 - r_1} = (-k)^n J(k,m-n). \qquad \square
\end{aligned}$$

Corollary 3.5. *For $m \geq n$ we have*

$$J_m J_{n+1} - J_{m+1}J_n = (-1)^n 2^n J_{m-n}.$$

Theorem 3.5. *Let $n, m \geq 1$, $m \geq n$, $k \geq 2$. Then*

$$j(k,m)j(k,n+1) - j(k,m+1)j(k,n) = (3-6k)(-k)^n J(k,m-n).$$

Proof. By (3.4) we have

$$\begin{aligned}
j(k,m)j(k,n+1) - j(k,m+1)(k,n) &= (c_1r_1^m + c_2r_2^m)(c_1r_1^{n+1} + c_2r_2^{n+1})\\
&\quad - (c_1r_1^{m+1} + c_2r_2^{m+1})(c_1r_1^n + c_2r_2^n)\\
&= c_1c_2(r_1^m r_2^{n+1} + r_1^{n+1}r_2^m - r_1^{m+1}r_2^n - r_1^n r_2^{m+1})\\
&= c_1c_2\big[r_1^m r_2^n(r_2-r_1) - r_1^n r_2^m(r_2-r_1)\big]\\
&= c_1c_2(r_2-r_1)(r_1^m r_2^n - r_1^n r_2^m)
\end{aligned}$$

$$= -c_1c_2(r_2 - r_1)(r_1r_2)^n(r_2^{m-n} - r_1^{m-n})$$

$$= -c_1c_2(r_2 - r_1)^2(r_1r_2)^n \cdot \frac{r_2^{m-n} - r_1^{m-n}}{r_2 - r_1}$$

$$= -(6k-3)(-k)^n J(k, m-n). \qquad \square$$

The next theorems present summation formulas for the presented numbers.

Theorem 3.6. *Let $n \geq 1$, $k \geq 2$ be integers. Then*

$$\sum_{i=0}^{n} J(k,i) = \frac{J(k,n+1) + kJ(k,n) - 1}{2k-2}, \tag{3.6}$$

$$\sum_{i=0}^{n} j(k,i) = \frac{j(k,n+1) + kj(k,n) + 2k - 5}{2k-2}. \tag{3.7}$$

Proof. Using the formula (2.8), we obtain

$$\sum_{i=0}^{n} J(k,i) = \sum_{i=0}^{n}(C_1r_1^i + C_2r_2^i) = C_1\frac{1-r_1^{n+1}}{1-r_1} + C_2\frac{1-r_2^{n+1}}{1-r_2}$$

$$= \frac{C_1 + C_2 - (C_1r_2 + C_2r_1) - (C_1r_1^{n+1} + C_2r_2^{n+1}) + r_1r_2(C_1r_1^n + C_2r_2^n)}{1 - (r_1 + r_2) + r_1r_2}$$

$$= \frac{C_1 + C_2 - (C_1r_2 + C_2r_1) - J(k,n+1) + r_1r_2J(k,n)}{1 - (r_1 + r_2) + r_1r_2}.$$

By simple calculations we obtain

$$C_1r_2 + C_2r_1 = -1. \tag{3.8}$$

By (3.8), (2.7) and (2.5) we get

$$\sum_{i=0}^{n} J(k,i) = \frac{J(k,n+1) + kJ(k,n) - 1}{2k-2}.$$

By formulas (2.9), (2.5) and (2.7) we have

$$\sum_{i=0}^{n} j(k,i) = \sum_{i=0}^{n}(c_1r_1^i + c_2r_2^i) = c_1\frac{1-r_1^{n+1}}{1-r_1} + c_2\frac{1-r_2^{n+1}}{1-r_2}$$

$$= \frac{c_1 + c_2 - (c_1r_2 + c_2r_1) - (c_1r_1^{n+1} + c_2r_2^{n+1}) - k(c_1r_1^n + c_2r_2^n)}{1 - (k-1) - k}.$$

Thus

$$\sum_{i=0}^{n} j(k,i) = \frac{c_1 + c_2 - (c_1 r_2 + c_2 r_1) - j(k,n+1) - kj(k,n)}{2-2k}.$$

Using the fact that $c_1 + c_2 = 2$ and

$$c_1 r_2 + c_2 r_1 = \frac{2k^2 - k - 3}{k+1} = 2k - 3$$

we get the equality (3.7). $\square$

Corollary 3.6. *Let $n \in \mathbb{N}$. Then*

$$\sum_{i=0}^{n} J_i = \frac{J_{n+2} - 1}{2},$$

$$\sum_{i=0}^{n} j_i = \frac{j_{n+2} - 1}{2}.$$

Theorem 3.7. *The generating function of the sequence $\{J(k,n)\}$ has the following form*

$$g(x) = \frac{x}{1-(k-1)x-kx^2}.$$

Proof. Let $g(x) = \sum\limits_{n=0}^{\infty} J(k,n)x^n$. Then

$$\begin{aligned}(1-(k-1)x-kx^2)g(x) &= (1-(k-1)x-kx^2)\sum_{n=0}^{\infty} J(k,n)x^n \\ &= \sum_{n=0}^{\infty} J(k,n)x^n - (k-1)\sum_{n=0}^{\infty} J(k,n)x^{n+1} - k\sum_{n=0}^{\infty} J(k,n)x^{n+2} \\ &= \sum_{n=2}^{\infty}(J(k,n)-(k-1)J(k,n-1)-kJ(k,n-2))x^n \\ &\quad + (J(k,0)+J(k,1)x)-(k-1)J(k,0)x.\end{aligned}$$

Using recurrence (2.1) we have

$$(1-(k-1)x-kx^2)g(x) = x.$$

Hence

$$g(x)=\frac{x}{1-(k-1)x-kx^2},$$

which completes the proof. □

Similarly we get the following result for $\{j(k,n)\}$.

Theorem 3.8. *The generating function of the sequence $\{j(k,n)\}$ is*

$$G(x)=\frac{2+(3-2k)x}{1-(k-1)x-kx^2}.$$

In the end we give matrix representations of the numbers $J(k,n)$ and $j(k,n)$.

Theorem 3.9. *Let $n\geq 1$, $k\geq 2$. Then*

$$\begin{bmatrix} J(k,n+1) & J(k,n) \\ J(k,n) & J(k,n-1) \end{bmatrix}=\begin{bmatrix} J(k,2) & J(k,1) \\ J(k,1) & J(k,0) \end{bmatrix}\cdot\begin{bmatrix} k-1 & 1 \\ k & 0 \end{bmatrix}^{n-1}. \tag{3.9}$$

Proof. We use induction on n. If $n=1$ then the result is obvious. Assuming the formula (3.9) holds for $n\geq 1$, we shall prove it for $n+1$.

Using induction's hypothesis and formula (2.1), we have

$$\begin{aligned}
&\begin{bmatrix} J(k,2) & J(k,1) \\ J(k,1) & J(k,0) \end{bmatrix}\cdot\begin{bmatrix} k-1 & 1 \\ k & 0 \end{bmatrix}^{n-1}\cdot\begin{bmatrix} k-1 & 1 \\ k & 0 \end{bmatrix}\\
&\qquad=\begin{bmatrix} J(k,n+1) & J(k,n) \\ J(k,n) & J(k,n-1) \end{bmatrix}\cdot\begin{bmatrix} k-1 & 1 \\ k & 0 \end{bmatrix}\\
&\qquad=\begin{bmatrix} (k-1)J(k,n+1)+kJ(k,n) & J(k,n+1) \\ (k-1)J(k,n)+kJ(k,n-1) & J(k,n) \end{bmatrix}\\
&\qquad=\begin{bmatrix} J(k,n+2) & J(k,n+1) \\ J(k,n+1) & J(k,n) \end{bmatrix},
\end{aligned}$$

which ends the proof. □

As a consequence of Theorem 3.9 we obtain Cassini's identity (3.3).

Corollary 3.7. *For $n\in\mathbb{N}$ we have*

$$J(k,n+1)J(k,n-1)-J^2(k,n)=(-1)^n k^{n-1}.$$

Proof. Calculating determinants in formula (3.9), we obtain

$$\begin{vmatrix} J(k,n+1) & J(k,n) \\ J(k,n) & J(k,n-1) \end{vmatrix} = J(k,n+1)J(k,n-1) - J^2(k,n),$$

$$\begin{vmatrix} J(k,2) & J(k,1) \\ J(k,1) & J(k,0) \end{vmatrix} = \begin{vmatrix} k-1 & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

$$\begin{vmatrix} k-1 & 1 \\ k & 0 \end{vmatrix} = -k.$$

By (3.9) we get

$$J(k,n+1)J(k,n-1) - J^2(k,n) = -(-k)^{n-1} = (-1)^n k^{n-1},$$

which completes the proof. □

Similarly to Theorem 3.9 and Corollary 3.7 we can get the next results.

Theorem 3.10. *Assume that $n \geq 1$, $k \geq 2$ are integers. Then*

$$\begin{bmatrix} j(k,n+1) & j(k,n) \\ j(k,n) & j(k,n-1) \end{bmatrix} = \begin{bmatrix} j(k,2) & j(k,1) \\ j(k,1) & j(k,0) \end{bmatrix} \cdot \begin{bmatrix} k-1 & 1 \\ k & 0 \end{bmatrix}^{n-1}.$$

Corollary 3.8. *For $n \in \mathbb{N}$ we have*

$$j(k,n+1)j(k,n-1) - j^2(k,n) = (6k-3)(-1)^{n-1}k^{n-1}.$$

4. Conclusions

Jacobsthal numbers belong to the family of the Fibonacci type numbers, i.e. numbers which are defined by the homogenous linear recurrence with constant coefficients. Such numbers are often extended to the negative domain, too. Our results obtained for generalized Jacobsthal numbers and generalized Jacobsthal–Lucas numbers may be a contribution to considerations about different one parameter or two parameters generalizations of the Jacobsthal numbers with negative index terms.

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