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THE COSINE-SINE FUNCTIONAL EQUATION ON SEMIGROUPS

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Abstract. The primary object of study is the "cosine-sine" functional equation $f(xy) = f(x)g(y)+g(x)f(y)+h(x)h(y)$ for unknown functions $f, g, h\colon S \to \mathbb{C}$, where S is a semigroup. The name refers to the fact that it contains both the sine and cosine addition laws. This equation has been solved on groups and on semigroups generated by their squares. Here we find the solutions on a larger class of semigroups and discuss the obstacles to finding a general solution for all semigroups. Examples are given to illustrate both the results and the obstacles.

We also discuss the special case $f(xy) = f(x)g(y) + g(x)f(y) - g(x)g(y)$ separately, since it has an independent direct solution on a general semigroup.

We give the continuous solutions on topological semigroups for both equations.

1. Introduction

Let S be a semigroup. The *cosine-sine* functional equation is

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad \text{for all } x, y \in S, \tag{1.1}$$

where $f, g, h\colon S \to \mathbb{C}$. This equation generalizes both the sine addition formula ($h = 0$) and the cosine addition formula ($g = \frac{1}{2}f$). Equation (1.1) was solved

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by Chung, Kannappan, and Ng ([1]) for the case that S is a group. Their result was extended by the author ([2]) to the case that S is a semigroup generated by its squares. The main goal of the present work is to extend that result to a larger class of semigroups. We also discuss the obstacles to finding the solution on a general semigroup.

We treat separately the special case $h = ig$ of (1.1), namely

$$f(xy) = f(x)g(y) + g(x)f(y) - g(x)g(y), \quad \text{for all } x, y \in S. \tag{1.2}$$

The solutions of (1.2) are described by Stetkær ([6]) in terms of exponentials and solutions of the sine addition formula. Using the recent solution of the sine addition formula by the author, we flesh out that description. We arrive at the solutions of (1.2) by this route rather than as a corollary of our result about (1.1), for two reasons. The first is because the method in [6] is direct and elementary, and the second is that (1.2) is solved on a general semigroup (with no extra conditions).

The functional equations above are of the Levi-Civita type, which includes all functional equations of the form

$$f(xy) = \sum_{k=1}^{n} g_k(x)h_k(y), \quad \text{for all } x, y \in S,$$

for unknown functions $f, g_k, h_k \colon S \to \mathbb{C}$ and any positive integer n. If S is an Abelian group and $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ are linearly independent, then it is known (see [7, Theorem 10.4]) that all solutions of such equations are exponential polynomials. An exponential polynomial is a linear combination of exponential monomials, which are terms of the form $(A_1)^{n_1} \cdots (A_k)^{n_k}\chi$ with χ exponential, each A_ℓ additive, and each n_ℓ a nonnegative integer.

The authors of [1] showed that the solutions of (1.1) on any group are exponential polynomials, whereas in [2] we showed that this is not generally the case on semigroups. For example, where a solution of (1.1) on a group contains a term $A\chi$, with A additive and χ exponential, on a semigroup we may see instead the term

$$\varphi(x) = \begin{cases} A(x)\chi(x) & \text{if } \chi(x) \neq 0, \\ 0 & \text{if } \chi(x) = 0 \end{cases}$$

(or even more complicated, see Proposition 2.1), where A is an additive function defined on the subsemigroup where χ is nonzero. Such a function φ need not be an exponential polynomial (for more see [4]). So there is an increase in complexity of solution forms of Levi-Civita equations as we move from the world of groups to the larger world of semigroups. To solve (1.1) we will impose certain conditions (introduced just after Lemma 4.1) on S that enable us to maintain some control over this increased complexity. It should be noted that every group satisfies the conditions we shall impose on S.

The outline of the paper is as follows. The next section introduces some notation, terminology, and the solution of the sine addition formula on semigroups. In the short section 3 we combine Stetkær's result about (1.2) with the general solution of the sine addition formula to get a more complete picture of the solutions of (1.2). Section 4 contains preparations for our primary objective, namely the solution of (1.1) on semigroups satisfying certain conditions. The solution is given in Theorem 5.1. The final section contains some examples applying our results about (1.2) and (1.1), and an example illustrating the complications that can arise when trying to solve (1.1) on a semigroup not satisfying the conditions in Theorem 5.1.

2. Notation, terminology, and preliminaries

Throughout this paper S denotes a semigroup. If S is a topological semigroup, $C(S)$ denotes the algebra of continuous functions from S into $\mathbb{C}$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

For any subset $T \subseteq S$ define $T^2 := \{t_1 t_2 \mid t_1, t_2 \in T\}$, so the notation T^2 will not be used to denote the direct product $T \times T$ in this article.

A function $A\colon S \to \mathbb{C}$ is *additive* if $A(xy) = A(x) + A(y)$ for all $x, y \in S$.

A function $\chi\colon S \to \mathbb{C}$ is *multiplicative* if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$. A multiplicative $\chi \neq 0$ is called an *exponential* on S. Unlike the situation on groups, an exponential on a semigroup can take the value 0 on a non-empty proper subset. We define the *nullspace* of an exponential $\chi\colon S \to \mathbb{C}$ by

$$I_\chi := \{x \in S \mid \chi(x) = 0\}.$$

If $I_\chi \neq \emptyset$ then it is an ideal (two-sided) of S. In fact it is a *prime ideal*, meaning that $I_\chi \neq S$ and $S \setminus I_\chi$ is a subsemigroup of S. There is an intimate connection between exponentials and prime ideals. We have already noted that if $\chi\colon S \to \mathbb{C}$ is exponential and $I_\chi \neq \emptyset$ then I_χ is a prime ideal. Conversely, if I is a prime ideal of S then there exists an exponential $\chi\colon S \to \mathbb{C}$ such that $I_\chi = I$, namely let $\chi(x) = 1$ for $x \in S \setminus I$ and $\chi(x) = 0$ for $x \in I$.

In order to describe some of our solutions on semigroups we partition the nullspace into the disjoint union $I_\chi = I_\chi^2 \cup P_\chi^{(1)} \cup P_\chi^{(1+)}$, where

$$P_\chi^{(1)} := \{p \in I_\chi \setminus I_\chi^2 \mid \text{ for all } w \in S \setminus I_\chi \text{ we have } pw \in I_\chi \setminus I_\chi^2\},$$

$$P_\chi^{(1+)} := \{p \in I_\chi \setminus I_\chi^2 \mid \text{ there exists } w_p \in S \setminus I_\chi \text{ such that } pw_p \in I_\chi^2\}.$$

(Note that $pw \in S \setminus I_\chi$ is impossible for $p \in I_\chi$ since I_χ is an ideal if nonempty.)

A function $F\colon S \to \mathbb{C}$ is *Abelian* if for every $n \geq 2$, permutation π on $\{1, \dots, n\}$, and $x_1, \dots, x_n \in S$ we have $F(x_{\pi(1)} \cdots x_{\pi(n)}) = F(x_1 \cdots x_n)$. Note that all additive functions and multiplicative functions are Abelian.

Define the relation $\sim$ on a semigroup S by $x \sim y$ if and only if there exist $s_1, \dots, s_n \in S$ and a permutation π on $\{1, \dots, n\}$ such that $x = s_1 \cdots s_n$ and $y = s_{\pi(1)} \cdots s_{\pi(n)}$. It is clear that if $x \sim y$ then $F(x) = F(y)$ for any Abelian function $F\colon S \to \mathbb{C}$. We read the statement $x \sim y$ as "*x rearranges to y*."

The following proposition is [3, Theorem 2.1]. The description of h in part (iii) gives an indication of the additional complexity of solutions of Levi-Civita functional equations on semigroups as opposed to groups.

Proposition 2.1. *Let S be a semigroup, and suppose $h, g\colon S \to \mathbb{C}$ satisfy the sine addition law*

$$h(xy) = h(x)g(y) + g(x)h(y), \quad x, y \in S, \tag{2.1}$$

with $h \neq 0$. Then h and g are Abelian and there exist multiplicative functions $\chi_1, \chi_2\colon S \to \mathbb{C}$ such that $g = \frac{\chi_1 + \chi_2}{2}$. In addition we have the following.

(i) *For $\chi_1 \neq \chi_2$ we have $h = c(\chi_1 - \chi_2)$ for some constant $c \in \mathbb{C}^*$.*

(ii) *For $\chi_1 = \chi_2 = 0 = g$ we have $S \neq S^2$ and*

$$h(x) = \begin{cases} h_0(x) & \text{for } x \in S \setminus S^2, \\ 0 & \text{for } x \in S^2, \end{cases}$$

where $h_0\colon S \setminus S^2 \to \mathbb{C}$ is an arbitrary nonzero function.

(iii) *For $\chi_1 = \chi_2 =: \chi \neq 0$ we have $g = \chi$, and h has the form*

$$h(x) = \begin{cases} A(x)\chi(x) & \text{for } x \in S \setminus I_\chi, \\ \rho(x) & \text{for } x \in P_\chi^{(1)}, \\ 0 & \text{for } x \in I_\chi^2 \cup P_\chi^{(1+)}, \end{cases}$$

where $A\colon S \setminus I_\chi \to \mathbb{C}$ is additive, ρ is the restriction of h to $P_\chi^{(1)}$, and the following two conditions hold.

(I) *If* $x \sim pw$ *with* $p \in P_\chi^{(1+)}$ *and* $w \in S \setminus I_\chi$, *then* $h(x) = 0$.

(II) *If* $x = pw$ *with* $p \in P_\chi^{(1)}$ *and* $w \in S \setminus I_\chi$, *then* $x \in P_\chi^{(1)}$ *and* $\rho(x) = \rho(p)\chi(w)$.

Note the possibility that some values of ρ *may be chosen arbitrarily.*

The converse statements are also true if $h \neq 0$ *in part* (iii).

Furthermore, if S *is a topological semigroup and* $h \in C(S)$, *then* $g, \chi_1, \chi_2, \chi \in C(S)$, $A \in C(S \setminus I_\chi)$, *and* $\rho \in C(P_\chi^{(1)})$.

The function ρ in part (iii) can take arbitrary values at some, none, or all points of $P_\chi^{(1)}$, as demonstrated by examples in [3].

Notation 2.2. Let $\Phi_{A,\chi,\rho}\colon S \to \mathbb{C}$ denote a function having the form of h in part (iii) of Proposition 2.1, where $\chi\colon S \to \mathbb{C}$ is an exponential, $A\colon S \setminus I_\chi \to \mathbb{C}$ is additive, ρ is the restriction of h to $P_\chi^{(1)}$, and conditions (I) and (II) hold.

Note that if S has no prime ideals (for instance if S is a group) then $\Phi_{A,\chi,\rho} = A\chi$.

3. The solution of (1.2)

We treat equation (1.2) first, since the solution is found directly (*i.e.* without reference to (1.1)) on a general semigroup. The next result is [6, Theorem 5.1], modified slightly to eliminate an overlap between cases (c) and (d).

Proposition 3.1. *Let* S *be a semigroup. The solutions* $f, g\colon S \to \mathbb{C}$ *of* (1.2) *are the following pairs of Abelian functions, where* $\chi, \chi_1, \chi_2\colon S \to \mathbb{C}$ *are exponentials such that* $\chi_1 \neq \chi_2$, $\phi\colon S \to \mathbb{C}$ *is a solution of the (special) sine addition formula* $\phi(xy) = \phi(x)\chi(y) + \chi(x)\phi(y)$ *such that* $\phi \neq 0$, $\alpha \in \mathbb{C} \setminus \{0, \frac{1}{2}\}$, *and* $\beta \in \mathbb{C} \setminus \{0, \pm 1\}$.

(a) f *is any function such that* $f(S^2) = \{0\}$, *and* $g = 0$.

(b) f *is any nonzero function such that* $f(S^2) = \{0\}$, *and* $g = 2f$.

(c)
$$f = \frac{\alpha^2}{2\alpha - 1}\chi \quad \text{and} \quad g = \alpha\chi.$$

(d)
$$f = \frac{1}{2}(\chi_1 + \chi_2) + \frac{\beta^2 + 1}{4\beta}(\chi_1 - \chi_2) \quad \text{and} \quad g = \frac{1}{2}(\chi_1 + \chi_2) + \frac{\beta}{2}(\chi_1 - \chi_2).$$

(e)
$$f = \phi + \chi \quad \textit{and} \quad g = \chi.$$

(f)
$$f = \frac{1}{2}\phi + \chi \quad \textit{and} \quad g = \phi + \chi.$$

All of the following except the topological part follows directly from Propositions 3.1 and 2.1.

Corollary 3.2. *Let S be a semigroup. The solutions $f, g\colon S \to \mathbb{C}$ of (1.2) are the following pairs of Abelian functions, where $\chi, \chi_1, \chi_2\colon S \to \mathbb{C}$ are exponentials such that $\chi_1 \neq \chi_2$, $\alpha \in \mathbb{C} \setminus \{0, \frac{1}{2}\}$, and $\beta \in \mathbb{C} \setminus \{0, \pm 1\}$.*

(a) *f is any function such that $f(S^2) = \{0\}$, and $g = 0$.*

(b) *f is any nonzero function such that $f(S^2) = \{0\}$, and $g = 2f$.*

(c)
$$f = \frac{\alpha^2}{2\alpha - 1}\chi \quad \textit{and} \quad g = \alpha\chi.$$

(d)
$$f = \frac{1}{2}(\chi_1 + \chi_2) + \frac{\beta^2 + 1}{4\beta}(\chi_1 - \chi_2) \quad \textit{and} \quad g = \frac{1}{2}(\chi_1 + \chi_2) + \frac{\beta}{2}(\chi_1 - \chi_2).$$

(e) *For some $\Phi_{A,\chi,\rho} \neq 0$ we have*
$$f = \Phi_{A,\chi,\rho} + \chi \quad \textit{and} \quad g = \chi.$$

(f) *For some $\Phi_{A,\chi,\rho} \neq 0$ we have*
$$f = \frac{1}{2}\Phi_{A,\chi,\rho} + \chi \quad \textit{and} \quad g = \Phi_{A,\chi,\rho} + \chi.$$

Furthermore, if S is a topological semigroup and $f, g \in C(S)$, then $\chi, \chi_1, \chi_2 \in C(S)$, $A \in C(S \setminus I_\chi)$, and $\rho \in C(P_\chi^{(1)})$.

Proof. The only things needing proof are the topological statements for parts (d), (e) and (f). For part (d) we have
$$g - f = \frac{\beta^2 - 1}{4\beta}(\chi_1 - \chi_2).$$

Since $\beta^2 \neq 1$ and $\chi_1 \neq \chi_2$, the continuity of χ_1 and χ_2 follows from [5, Theorem 3.18].

In part (e) we get immediately that χ and $\Phi_{A,\chi,\rho}$ are continuous. By definition this yields that $x \mapsto A\chi(x)$ is continuous on $S \setminus I_\chi$ and $\rho \in C(P_\chi^{(1)})$. Since $\chi(x) \neq 0$ for $x \in S \setminus I_\chi$ we have $A \in C(S \setminus I_\chi)$.

Finally, in case (f) we see that $2(g - f) = \Phi_{A,\chi,\rho}$ is continuous, thus $\chi \in C(S)$ and the rest follows as before. □

Some examples are given in the final section.

4. Preparations for the solution of (1.1)

Now we turn toward our primary goal of solving (1.1). The following is a part of [1, Lemma 4]. It is stated for groups but the same proof works for semigroups.

Lemma 4.1. *If $f, h\colon S \to \mathbb{C}$ satisfy*

$$f(xy) = f(x)f(y) + h(x)h(y), \quad x, y \in S,$$

then there exists a constant $\alpha \in \mathbb{C}$ such that

$$h(xy) = h(x)f(y) + f(x)h(y) + \alpha h(x)h(y), \quad x, y \in S.$$

For the consideration of (1.1) we shall impose the following conditions on our semigroup S.

Definition 4.2. Let S be a semigroup. We will say that S is *compatible*, if $S = S^2$ and for every prime ideal $I \subset S$ the following condition holds.

(4.1) For each $q \in I$ there exists $w_q \in S \setminus I$ such that $qw_q \in I^2$.

We say that a topological semigroup S is “topologically compatible”, or *t-compatible*, if $S = S^2$ and condition (4.1) holds for every prime ideal I serving as the null ideal of an exponential in $C(S)$.

Note that $S = S^2$ is satisfied for instance by every monoid (*i.e.* a semigroup with identity element). Condition (4.1) is satisfied for example by semigroups with no prime ideals, and by semigroups in which $I = I^2$ for every prime ideal I.

The next lemma shows that the solution of (1.1) on compatible semigroups will generalize the results of [1] and [2].

Lemma 4.3. *The class consisting of groups and semigroups generated by their squares is a proper subset of the class of compatible semigroups.*

Proof. Groups satisfy $S = S^2$ since they have an identity element, and they trivially satisfy (4.1) since they have no prime ideals. Suppose S is a semigroup generated by its squares and $I \subset S$ is a prime ideal. The proofs that $S = S^2$ and $I = I^2$ are similar, so we prove only the second one. For any $x \in I$ there exist a positive integer n and $y_1, \dots, y_n \in S$ such that $x = y_1^2 \cdots y_n^2$, since S is generated by its squares. Since I is a prime ideal we have $y_j \in I$ for some $1 \le j \le n$, therefore $x = (y_1^2 \cdots y_{j-1}^2 y_j)(y_j y_{j+1}^2 \cdots y_n^2) \in I^2$. Thus $I = I^2$ and S satisfies (4.1). This proves that groups and semigroups generated by their squares are compatible semigroups.

On the other hand, the semigroup $S = (-1, 1)$ under multiplication is not a group and is not generated by its squares. Clearly $S = S^2$. The only prime ideal of S is $I = \{0\}$, and $I = I^2$. Thus S satisfies condition (4.1) and is therefore compatible. □

By definition, if S is a compatible semigroup then $P_\chi^{(1)} = \emptyset$ for every multiplicative $\chi\colon S \to \mathbb{C}$. In such an event the form of h in Proposition 2.1(iii) simplifies to

$$h(x) = \begin{cases} A(x)\chi(x) & \text{for } x \in S \setminus I_\chi, \\ 0 & \text{for } x \in I_\chi. \end{cases} \tag{4.2}$$

Notation 4.4. Let $\Phi_{A,\chi}\colon S \to \mathbb{C}$ denote a function h of the form (4.2), where $\chi\colon S \to \mathbb{C}$ is an exponential and $A\colon S \setminus I_\chi \to \mathbb{C}$ is additive.

From this point on we will generally state results in their topological versions. One can get algebraic (non-topological) versions by taking the discrete topology. We have the following corollary of Proposition 2.1.

Corollary 4.5. *Let S be a t-compatible topological semigroup. If $h, g \in C(S)$ satisfy the sine addition law (2.1) with $h \neq 0$, then h, g belong to one of the following families, where $\chi_1, \chi_2, \chi \in C(S)$ are multiplicative, $A \in C(S \setminus I_\chi)$ is a nonzero additive function, and $c \in \mathbb{C}^*$.*

(i) *For $\chi_1 \neq \chi_2$ we have*

$$h = c(\chi_1 - \chi_2) \quad \text{and} \quad g = \frac{\chi_1 + \chi_2}{2}.$$

(ii) *For $g = \chi \neq 0$ we have $h = \Phi_{A,\chi}$ as described in Notation 4.4.*

The converse statements are also true.

Proof. Part (i) carries over from Proposition 2.1, but Proposition 2.1(ii) is nullified by the imposed condition $S = S^2$. Proposition 2.1(iii) carries over to the current case (ii) since $P_\chi^{(1)} = \emptyset$. Furthermore $A \neq 0$ because $h \neq 0$. The converse is easily verified. □

Next we consider a system of functional equations arising in the process of solving (1.1).

Lemma 4.6. *Let S be a t-compatible topological semigroup, and suppose $h \in C(S)$ has the form $h = \Phi_{A,\chi} \neq 0$ with exponential $\chi \in C(S)$. If $f \in C(S)$ satisfies*

$$f(xy) = f(x)\chi(y) + \chi(x)f(y) + h(x)h(y), \quad x, y \in S, \tag{4.3}$$

then

$$f(x) = \begin{cases} \left(A'(x) + \frac{1}{2}A(x)^2\right)\chi(x) & \text{for } x \in S \setminus I_\chi, \\ 0 & \text{for } x \in I_\chi, \end{cases} \tag{4.4}$$

where $A' \in C(S \setminus I_\chi)$ is additive.

Conversely, if f is given by (4.4) *with additive $A' \colon S \setminus I_\chi \to \mathbb{C}$ and $h = \Phi_{A,\chi}$, then* (4.3) *holds.*

Proof. Suppose f, h satisfy (4.3) with $h = \Phi_{A,\chi} \neq 0$. For $x, y \in S \setminus I_\chi$, dividing (4.3) by $\chi(x)\chi(y)$ we get

$$\frac{f}{\chi}(xy) = \frac{f}{\chi}(x) + \frac{f}{\chi}(y) + A(x)A(y), \quad x, y \in S \setminus I_\chi.$$

Thus $(f/\chi) - \frac{1}{2}A^2 =: A' \in C(S \setminus I_\chi)$ is additive, and we have the top case of (4.4). If $I_\chi = \emptyset$ then we are done (with the bottom case of (4.4) vacuous). If $I_\chi \neq \emptyset$ and $x, y \in I_\chi$, then (4.3) yields $f(xy) = 0$ since $\chi(x) = \chi(y) = h(x) = 0$. Thus f vanishes on I_χ^2. For any $x \in I_\chi$, by t-compatibility there exists $w_x \in S \setminus I_\chi$ such that $xw_x \in I^2$. Thus by (4.3) we get

$$0 = f(xw_x) = f(x)\chi(w_x) + \chi(x)f(w_x) + h(x)h(w_x) = f(x)\chi(w_x)$$

since $\chi(x) = h(x) = 0$. Now $\chi(w_x) \neq 0$ implies $f(x) = 0$ and we have (4.4).

The converse is a simple verification. □

We introduce notation for the solution type of f above.

Notation 4.7. Let $\Psi_{A',A,\chi} \colon S \to \mathbb{C}$ denote a function f of the form (4.4), where $\chi \colon S \to \mathbb{C}$ is multiplicative and $A, A' \colon S \setminus I_\chi \to \mathbb{C}$ are additive.

Thus Lemma 4.6 shows that the pair $(f,h) = (\Psi_{A',A,\chi}, \Phi_{A,\chi})$ satisfies (4.3).

Now we gather some linear independence results for typical solution functions.

LEMMA 4.8. *Let S be a semigroup, and let $n \in \mathbb{N}$. Suppose $\chi, \chi', \chi_1, \chi_2, \dots, \chi_n \colon S \to \mathbb{C}$ are distinct exponentials, $A', A \colon S \setminus I_\chi \to \mathbb{C}$ are additive, and $\Phi_{A,\chi}, \Psi_{A',A,\chi} \colon S \to \mathbb{C}$ are as defined above.*

(a) *$\{\chi_1, \chi_2, \dots \chi_n\}$ is linearly independent.*
(b) *If $A \neq 0$ then $\{\chi, \Phi_{A,\chi}\}$ is linearly independent.*
(c) *If $A \neq 0$ then $\{\chi', \chi, \Phi_{A,\chi}\}$ is linearly independent.*
(d) *If $A \neq 0$ then $\{\chi, \Phi_{A,\chi}, \Psi_{A',A,\chi}\}$ is linearly independent.*

PROOF. Part (a) is part of [5, Theorem 3.18].

For part (b) suppose

$$a\chi + b\Phi_{A,\chi} = 0$$

for some constants $a, b \in \mathbb{C}$. Restricting to the subsemigroup $S \setminus I_\chi$ we get

$$a + bA(x) = 0, \quad x \in S \setminus I_\chi.$$

Thus $a = bA = 0$, so $b = 0$ since $A \neq 0$.

For part (c) suppose

$$a\chi' + b\chi + c\Phi_{A,\chi} = 0 \tag{4.5}$$

for some constants $a, b, c \in \mathbb{C}$. Then

$$am(x) + b + cA(x) = 0, \quad \text{for } x \in S \setminus I_\chi, \tag{4.6}$$

where $m \colon S \setminus I_\chi \to \mathbb{C}$ defined by $m := \chi'/\chi$ is multiplicative. Using (4.6) several times we find that

$$\begin{aligned} 0 &= am(xy) + b + cA(xy) = am(x)m(y) + b + cA(x) + cA(y) \\ &= a[m(x)m(y) - m(x) - m(y)] - b \\ &= a[m(x) - 1][m(y) - 1] - (a + b), \end{aligned}$$

so

$$a[m(x) - 1][m(y) - 1] = a + b, \quad \text{for all } x, y \in S \setminus I_\chi.$$

If $a \neq 0$ then m is constant, say $m(x) = \mu$ for all $x \in S \setminus I_\chi$. Putting this into (4.6) we have $a\mu + b + cA = 0$, so as before we find that $a\mu + b = 0$ and

$c = 0$. Now (4.5) yields $a\chi' + b\chi = 0$, and by part (a) this is possible only if $a = b = 0$, a contradiction. Thus $a = 0$, and $b = c = 0$ follows by part (b).

Finally, for part (d) let

$$a\chi + b\Phi_{A,\chi} + c\Psi_{A',A,\chi} = 0$$

for some constants $a, b, c \in \mathbb{C}$. Restricting to the sub-semigroup $S \setminus I_\chi$ we have

$$a + bA(x) + c[A'(x) + \frac{1}{2}A^2(x)] = 0, \quad x \in S \setminus I_\chi. \tag{4.7}$$

Since $A \neq 0$, each of the terms a, bA, cA' in (4.7) is homogeneous of degree 0 or 1, while the term $\frac{c}{2}A^2$ is homogeneous of degree 2 if $c \neq 0$, hence $c = 0$. Now (4.7) reduces to $a + bA = 0$, and since $A \neq 0$ we get $a = b = 0$. □

The following will also play an important role in our solution of (1.1).

Lemma 4.9. *Let S be a t-compatible topological semigroup, and let $\chi \in C(S)$ be multiplicative. If $f, h \in C(S)$ satisfy the pair of functional equations* (4.3) *and*

$$h(xy) = h(x)\big(\chi + \frac{\delta}{2}h\big)(y) + \big(\chi + \frac{\delta}{2}h\big)(x)h(y), \quad x, y \in S, \tag{4.8}$$

for some $\delta \in \mathbb{C}$, with f and h linearly independent, then they belong to one of the following families, where $\chi' \in C(S)$ is multiplicative, $A, A' \in C(S \setminus I_\chi)$ are additive, and $c \in \mathbb{C}^*$.

(a) *For $\delta \neq 0$ we have $h = c(\chi - \chi')$ and $f = -ch + \Phi_{A,\chi}$ where $\chi \neq \chi'$ and $\Phi_{A,\chi} \neq 0$.*

(b) *For $\delta = 0$ we have $h = \Phi_{A,\chi} \neq 0$ and $f = \Psi_{A',A,\chi} \neq 0$ (so χ is an exponential).*

The converse is also true.

Proof. We start with the "if" part. Since f, h are linearly independent we have $f \neq 0$, $h \neq 0$. Note that (4.8) can be viewed as the sine addition formula (2.1) with $g := \chi + \frac{\delta}{2}h$. We divide the proof according to the cases (i) and (ii) of Corollary 4.5 for the solutions h of (4.8).

Case (i): In case (i) of Corollary 4.5 we have $h = c(\chi_1 - \chi_2)$ and $g = \frac{1}{2}(\chi_1 + \chi_2)$ for multiplicative $\chi_1, \chi_2 \in C(S)$ with $\chi_1 \neq \chi_2$ and $c \in \mathbb{C}^*$. Thus we get

$$0 = g - g = (\chi + \frac{\delta}{2}h) - \frac{1}{2}(\chi_1 + \chi_2) = \chi + \frac{c\delta - 1}{2}\chi_1 - \frac{c\delta + 1}{2}\chi_2.$$

By Lemma 4.8(a), this implies that χ is equal to either χ_1 or χ_2. Without loss of generality suppose $\chi_1 = \chi$ and $\delta = -\frac{1}{c} \neq 0$. Now (4.3) and (4.8), together with the independence of f and h, show that

$$(h - \delta f)(xy) = (h - \delta f)(x)\chi(y) + \chi(x)(h - \delta f)(y)$$

with $h - \delta f \neq 0$. Applying Corollary 4.5 to the function $h - \delta f$, we are in case (ii), so $\chi \neq 0$ and $h - \delta f = \Phi_{A,\chi}$ for additive $A \in C(S \setminus I_\chi)$. Since $\delta \neq 0$ we can solve for f here, and we get solution family (a) after relabeling.

Case (ii): From Corollary 4.5(ii) we get $g = \chi_1 = \chi_2 =: \chi' \neq 0$ and $h = \Phi_{A,\chi'}$ for some exponential $\chi' \in C(S)$ and additive $A \in C(S \setminus I_{\chi'})$ with $A \neq 0$. Thus we have $0 = g - g = \chi' - (\chi + \frac{\delta}{2}h)$, so

$$\frac{\delta}{2}h = \chi' - \chi.$$

Since $h = \Phi_{A,\chi'} \neq 0$, this contradicts Lemma 4.8(c) unless $\delta = 0$ and $\chi = \chi'$. Thus we have $h = \Phi_{A,\chi}$. Applying Lemma 4.6 to (4.3) we get that $f = \Psi_{A',A,\chi}$ for some additive $A' \in C(S \setminus I_\chi)$. Thus we have the solution forms in (b).

For the converse, the verifications of (4.3) and (4.8) are straightforward. The linear independence of f and h is confirmed by Lemma 4.8. □

Next we verify the solution families of (1.1) that will be found in our main result. The method of proof parallels [1, Lemmas 1, 3, and 6].

LEMMA 4.10. *Let S be a semigroup, let $\chi_j, \chi, \chi' \colon S \to \mathbb{C}$ be multiplicative functions, let $A, A' \colon S \setminus I_\chi \to \mathbb{C}$ be additive, and let $f, g, h \colon S \to \mathbb{C}$ be a solution of* (1.1) *such that f, g, h belong to one of the following linear spaces V.*
(a) *If $V = span\{\chi_1, \chi_2, \chi_3\}$, then there exist $a_j, b_j, c_j \in \mathbb{C}$ satisfying*

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{4.9}$$

such that

$$f = \sum_{i=1}^{3} a_i\chi_i, \quad g = \sum_{i=1}^{3} b_i\chi_i, \quad h = \sum_{i=1}^{3} c_i\chi_i. \tag{4.10}$$

(b) *If* $V = span\{\chi', \chi, \Phi_{A,\chi}\}$*, then there exist* $a_j, b_j, c_j \in \mathbb{C}$ *satisfying*

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & a_3 & 0 \end{pmatrix} \tag{4.11}$$

such that

$$\begin{cases} f = a_1\chi' + a_2\chi + a_3\Phi_{A,\chi}, \\ g = b_1\chi' + b_2\chi + b_3\Phi_{A,\chi}, \\ h = c_1\chi' + c_2\chi + c_3\Phi_{A,\chi}. \end{cases} \tag{4.12}$$

(c) *If* $V = span\{\chi, \Phi_{A,\chi}, \Psi_{A',A,\chi}\}$*, then there exist* $a_j, b_j, c_j \in \mathbb{C}$ *satisfying*

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} \tag{4.13}$$

such that

$$\begin{cases} f = a_1\chi + a_2\Phi_{A,\chi} + a_3\Psi_{A',A,\chi}, \\ g = b_1\chi + b_2\Phi_{A,\chi} + b_3\Psi_{A',A,\chi}, \\ h = c_1\chi + c_2\Phi_{A,\chi} + c_3\Psi_{A',A,\chi}. \end{cases} \tag{4.14}$$

Conversely, in each case if f, g, h *is such a linear combination with coefficients satisfying the stated condition, then the functions satisfy* (1.1)*. Moreover if any one of* f, g, h *is zero then the corresponding coefficients can be chosen to be zero.*

Proof. For part (a), let $E \subseteq \{1, 2, 3\}$ be chosen so that $\{\chi_i \mid i \in E\}$ is a basis for V. Then there is a unique representation of f, g, h in the form (4.10) with $a_k = b_k = c_k = 0$ for all $k \notin E$. With $f = \sum_{i \in E} a_i\chi_i$, $g = \sum_{i \in E} b_i\chi_i$, $h = \sum_{i \in E} c_i\chi_i$, the linear independence of $\{\chi_i \mid i \in E\}$ implies that (1.1) is satisfied if and only if the constants $\{a_i, b_i, c_i \mid i \in E\}$ satisfy

$$a_ib_j + b_ia_j + c_ic_j = \delta_{ij}a_i,$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \neq i$. Combining this with $a_k = b_k = c_k = 0$ for $k \notin E$ we have the constraint (4.9).

In part (b), define $\gamma_1 := \chi'$, $\gamma_2 := \chi$, $\gamma_3 := \Phi_{A,\chi}$, and let $E \subseteq \{1, 2, 3\}$ be chosen so that $\{\gamma_i \mid i \in E\}$ is a basis for V. As before there is a unique representation of f, g, h in the form (4.12) with $a_k = b_k = c_k = 0$ for all $k \notin E$. Inserting these forms into (1.1) we find after some rearrangement that

$$\begin{aligned}
a_3\Phi_{A,\chi}(xy) = {} & (2a_1b_1 + c_1^2 - a_1)\chi'(x)\chi'(y) + (2a_2b_2 + c_2^2 - a_2)\chi(x)\chi(y) \\
& + (a_1b_2 + b_1a_2 + c_1c_2)[\chi'(x)\chi(y) + \chi(x)\chi'(y)] \\
& + (2a_3b_3 + c_3^2)\Phi_{A,\chi}(x)\Phi_{A,\chi}(y) \\
& + (a_1b_3 + b_1a_3 + c_1c_3)[\chi'(x)\Phi_{A,\chi}(y) + \Phi_{A,\chi}(x)\chi'(y)] \\
& + (a_2b_3 + b_2a_3 + c_2c_3)[\chi(x)\Phi_{A,\chi}(y) + \Phi_{A,\chi}(x)\chi(y)].
\end{aligned}$$

Since $\Phi_{A,\chi}(xy) = \Phi_{A,\chi}(x)\chi(y) + \chi(x)\Phi_{A,\chi}(y)$ we can rewrite the preceding equation as

$$\begin{aligned}
0 = {} & (2a_1b_1 + c_1^2 - a_1)\chi'(x)\chi'(y) + (2a_2b_2 + c_2^2 - a_2)\chi(x)\chi(y) \\
& + (a_1b_2 + b_1a_2 + c_1c_2)[\chi'(x)\chi(y) + \chi(x)\chi'(y)] \\
& + (2a_3b_3 + c_3^2)\Phi_{A,\chi}(x)\Phi_{A,\chi}(y) \\
& + (a_1b_3 + b_1a_3 + c_1c_3)[\chi'(x)\Phi_{A,\chi}(y) + \Phi_{A,\chi}(x)\chi'(y)] \\
& + (a_2b_3 + b_2a_3 + c_2c_3 - a_3)[\chi(x)\Phi_{A,\chi}(y) + \Phi_{A,\chi}(x)\chi(y)].
\end{aligned}$$

By the linear independence of basis elements, the coefficients of all nonzero terms vanish. Combining this with $a_k = b_k = c_k = 0$ for all $k \notin E$, we have (4.11).

For part (c) define $\gamma_1 := \chi$, $\gamma_2 := \Phi_{A,\chi}$, $\gamma_3 := \Psi_{A',A,\chi}$ and let $E \subseteq \{1, 2, 3\}$ be chosen so that $\{\gamma_i \mid i \in E\}$ is a basis for V. Again there is a unique representation of f, g, h in the form (4.14) with $a_k = b_k = c_k = 0$ for all $k \notin E$. Here we find that (1.1) is satisfied if and only if

$$\begin{aligned}
& a_1\chi(xy) + a_2\Phi_{A,\chi}(xy) + a_3\Psi_{A',A,\chi}(xy) \\
& = (2a_1b_1 + c_1^2)\chi(x)\chi(y) + (2a_2b_2 + c_2^2)\Phi_{A,\chi}(x)\Phi_{A,\chi}(y) \\
& \quad + (2a_3b_3 + c_3^2)\Psi_{A',A,\chi}(x)\Psi_{A',A,\chi}(y) \\
& \quad + (a_1b_2 + b_1a_2 + c_1c_2)[\chi(x)\Phi_{A,\chi}(y) + \Phi_{A,\chi}(x)\chi(y)] \\
& \quad + (a_1b_3 + b_1a_3 + c_1c_3)[\chi(x)\Psi_{A',A,\chi}(y) + \Psi_{A',A,\chi}(x)\chi(y)] \\
& \quad + (a_2b_3 + b_2a_3 + c_2c_3)[\Phi_{A,\chi}(x)\Psi_{A',A,\chi}(y) + \Psi_{A',A,\chi}(x)\Phi_{A,\chi}(y)].
\end{aligned}$$

Since $\Psi_{A',A,\chi}(xy) = \Psi_{A',A,\chi}(x)\chi(y) + \chi(x)\Psi_{A',A,\chi}(y) + \Phi_{A,\chi}(y)\Phi_{A,\chi}(y)$ and $\Phi_{A,\chi}(xy) = \Phi_{A,\chi}(x)\chi(y) + \chi(x)\Phi_{A,\chi}(y)$, the preceding equation reduces to

$$\begin{aligned}
0 &= (2a_1b_1 + c_1^2 - a_1)\chi(x)\chi(y) + (2a_2b_2 + c_2^2 - a_3)\Phi_{A,\chi}(x)\Phi_{A,\chi}(y) \\
&\quad + (2a_3b_3 + c_3^2)\Psi_{A',A,\chi}(x)\Psi_{A',A,\chi}(y) \\
&\quad + (a_1b_2 + b_1a_2 + c_1c_2 - a_2)[\chi(x)\Phi_{A,\chi}(y) + \Phi_{A,\chi}(x)\chi(y)] \\
&\quad + (a_1b_3 + b_1a_3 + c_1c_3 - a_3)[\chi(x)\Psi_{A',A,\chi}(y) + \Psi_{A',A,\chi}(x)\chi(y)] \\
&\quad + (a_2b_3 + b_2a_3 + c_2c_3)[\Phi_{A,\chi}(x)\Psi_{A',A,\chi}(y) + \Psi_{A',A,\chi}(x)\Phi_{A,\chi}(y)].
\end{aligned}$$

As before, the independence of basis elements implies that the coefficients of all nonzero terms vanish, and we have the claimed (4.13).

The converse statements are easily verified by substitution. □

5. The solution of (1.1)

Now we are ready for the main result. Observe that if $f = 0$ in (1.1), then $h = 0$ and g is an arbitrary function. We omit this trivial case from our theorem.

We adhere closely to the plan of the proof used in [1, Theorem]. Much of that proof is repeated here, for two reasons. One reason is for completeness, but the larger reason is that we arrive at a solution family in part (c) that is stated more concisely than the one in [1]. We comment on that point again after the proof.

THEOREM 5.1. *Let S be a t-compatible topological semigroup, and suppose $f, g, h \in C(S)$ satisfy* (1.1) *with $f \neq 0$. Then f, g, h belong to one of the three families below. In each family we can choose a basis B for V and coefficients a_j, b_j, c_j so that the coefficients are equal to 0 for each term not appearing in B. In addition the functions $\chi_j, \chi, \chi', \Phi_{A,\chi}, \Psi_{A',A,\chi} \colon S \to \mathbb{C}$ that appear in B belong to $C(S)$, where χ, χ', χ_j are multiplicative and $A, A' \in C(S \setminus I_\chi)$ are additive.*

(a) *$f, g, h \in V = span\{\chi_1, \chi_2, \chi_3\}$, namely*

$$f = \sum_{j=1}^{3} a_j\chi_j, \quad g = \sum_{j=1}^{3} b_j\chi_j, \quad h = \sum_{j=1}^{3} c_j\chi_j$$

with $a_j, b_j, c_j \in \mathbb{C}$ satisfying (4.9).

(b) $f,g,h \in V = span\{\chi', \chi, \Phi_{A,\chi}\}$, *namely*

$$f = a_1\chi' + a_2\chi + a_3\Phi_{A,\chi}, \quad g = b_1\chi' + b_2\chi + b_3\Phi_{A,\chi},$$
$$h = c_1\chi' + c_2\chi + c_3\Phi_{A,\chi}$$

with $a_j, b_j, c_j \in \mathbb{C}$ *satisfying* (4.11).

(c) $f,g,h \in V = span\{\chi, \Phi_{A,\chi}, \Psi_{A',A,\chi}\}$, *namely*

$$f = a_1\chi + a_2\Phi_{A,\chi} + a_3\Psi_{A',A,\chi},$$
$$g = b_1\chi + b_2\Phi_{A,\chi} + b_3\Psi_{A',A,\chi},$$
$$h = c_1\chi + c_2\Phi_{A,\chi} + c_3\Psi_{A',A,\chi},$$

with $a_j, b_j, c_j \in \mathbb{C}$ *satisfying* (4.13).

Conversely, the functions in each family satisfy (1.1).

Proof. Let $f,g,h \in C(S)$ be a solution of (1.1) with $f \neq 0$, and suppose first that $\{f,h\}$ is linearly dependent. Then $h = \lambda f$ and (1.1) can be written as

$$f(xy) = f(x)k(y) + k(x)f(y), \quad x,y \in S,$$

where $k \in C(S)$ is defined by

$$k := g + \frac{1}{2}\lambda^2 f.$$

By Corollary 4.5 there are two solution families. In case (i) we have $f = c(\chi_1 - \chi_2)$ and $k = \frac{1}{2}(\chi_1 + \chi_2)$ for a pair of distinct multiplicative functions $\chi_1, \chi_2 \in C(S)$ and $c \in \mathbb{C}^*$. Thus we have $f,g,h \in span\{\chi_1,\chi_2\}$ and are in family (a). In case (ii) we get that $k = \chi$ is a (continuous) exponential and $f = \Phi_{A,\chi}$. Thus $f,g,h \in span\{\chi, \Phi_{A,\chi}\}$, giving a solution belonging to family (b). From here on we assume that $\{f,h\}$ is linearly independent, so $h \neq 0$. Comparing the results of computing $f((xy)z)$ and $f(x(yz))$ using (1.1), and using the linear independence of f and h, we obtain as in [1] the pair of functional equations

$$\begin{aligned} (5.1) \quad g(xy) &= g(x)g(y) + \alpha f(x)f(y) + \beta[f(x)h(y) + h(x)f(y)] \\ &\quad + \gamma h(x)h(y), \end{aligned}$$

$$\begin{aligned} (5.2) \quad h(xy) &= h(x)g(y) + g(x)h(y) + \beta f(x)f(y) + \gamma[f(x)h(y) + h(x)f(y)] \\ &\quad + \delta h(x)h(y), \end{aligned}$$

for some constants $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then computing $g(x(yz))$ and $g((xy)z)$ using equations (5.1), (5.2), and (1.1), a comparison of those results brings us (again using linear independence of $\{f, h\}$) to the conclusion that

$$\alpha + \beta\delta - \gamma^2 = 0. \tag{5.3}$$

Next, using (1.1), (5.1), and the linear independence of $\{f, h\}$ we find that the functional equation

$$(\lambda f + g)(xy) = (\lambda f + g)(x)(\lambda f + g)(y) + (\mu f + \nu h)(x)(\mu f + \nu h)(y) \tag{5.4}$$

holds if and only if the constants $\lambda, \mu, \nu \in \mathbb{C}$ satisfy

$$\lambda^2 + \mu^2 = \alpha, \quad \mu\nu = \beta, \quad \text{and } \nu^2 = \lambda + \gamma. \tag{5.5}$$

Now the proof divides into two main cases.

Case 1: Suppose $\beta = 0$. Now from (5.3) we have $\alpha = \gamma^2$, and the choice $(\lambda, \mu, \nu) = (-\gamma, 0, 0)$ yields a solution of (5.5). Therefore by (5.4) we have

$$g = \chi + \gamma f \tag{5.6}$$

for some multiplicative $\chi \in C(S)$. Using this to eliminate g from (1.1) and (5.2), we arrive at the pair of functional equations

$$f(xy) = 2\gamma f(x)f(y) + f(x)\chi(y) + \chi(x)f(y) + h(x)h(y), \tag{5.7}$$

$$h(xy) = h(x)\big(\chi + 2\gamma f + \frac{\delta}{2}h\big)(y) + \big(\chi + 2\gamma f + \frac{\delta}{2}h\big)(x)h(y). \tag{5.8}$$

Here we subdivide the proof again.

Subcase 1a: Suppose $\gamma = 0$. Now (5.7) and (5.8) reduce to

$$f(xy) = f(x)\chi(y) + \chi(x)f(y) + h(x)h(y),$$

$$h(xy) = h(x)\big(\chi + \frac{\delta}{2}h\big)(y) + \big(\chi + \frac{\delta}{2}h\big)(x)h(y),$$

with solutions given by Lemma 4.9. If $\delta \neq 0$ then we have $h = c(\chi - \chi')$ and $f = -ch + \Phi_{A,\chi}$ with $\chi, \chi', \Phi_{A,\chi} \in C(S)$, for $\chi \neq \chi'$ (multiplicative), $c \in \mathbb{C}^*$, and $\Phi_{A,\chi} \neq 0$. In this case by (5.6) we have $f, g, h \in span\{\chi', \chi, \Phi_{A,\chi}\}$ and are again in solution family (b). If $\delta = 0$ then we have $h = \Phi_{A,\chi} \neq 0$ and $f = \Psi_{A',A,\chi} \neq 0$. Thus by (5.6) we have $f, g, h \in span\{\chi, \Phi_{A,\chi}, \Psi_{A',A,\chi}\}$ and are in family (c).

Subcase 1b: Suppose $\gamma \neq 0$. Applying Corollary 4.5 to (5.8) yields two solution families for the pair h, k, where $k := \chi + 2\gamma f + \frac{\delta}{2}h \in C(S)$. The first family is $h = c(\chi_1 - \chi_2)$, $k = \frac{1}{2}(\chi_1 + \chi_2)$ for distinct multiplicative $\chi_1, \chi_2 \in C(S)$. Comparing the two equations for k, we see that $f \in span\{\chi_1, \chi_2, \chi\}$. Defining $\chi_3 := \chi$ and recalling (5.6), we have $f, g, h \in span\{\chi_1, \chi_2, \chi_3\}$ and are in family (a) again.

The second solution family of (5.8) from Corollary 4.5 is $h = \Phi_{A,\chi'}$, $k = \chi'$, for some exponential $\chi' \in C(S)$ and nonzero additive $A \in C(S \setminus I_{\chi'})$. Equating the two formulas for k here, we find that $f \in span\{\chi', \chi, \Phi_{A,\chi'}\}$. By (5.6) we have $f, g, h \in span\{\chi', \chi, \Phi_{A,\chi'}\}$ and (switching the roles of χ and χ') are in family (b) again. This completes Case 1.

Case 2: Suppose $\beta \neq 0$. Choosing constants λ, μ, ν satisfying (5.5) we rewrite (5.4) as

$$(\lambda f + g)(xy) = (\lambda f + g)(x)(\lambda f + g)(y) + \nu H(x)\nu H(y), \tag{5.9}$$

where $\mu\nu = \beta \neq 0$ and $H \in C(S)$ is defined by

$$\nu H := \mu f + \nu h. \tag{5.10}$$

From the independence of f, h we have $H \neq 0$, therefore $\lambda f + g \neq 0$. Applying Lemma 4.1 to (5.9) we get

$$\nu H(xy) = (\lambda f + g)(x)\nu H(y) + \nu H(x)(\lambda f + g)(y) + \eta\nu H(x)\nu H(y)$$

for some $\eta \in \mathbb{C}$, that is

$$H(xy) = H(x)(\lambda f + g + \frac{\eta\nu}{2}H)(y) + (\lambda f + g + \frac{\eta\nu}{2}H)(x)H(y). \tag{5.11}$$

Next we eliminate h from (1.1) using (5.10), resulting in

$$f(xy) = f(x)\big(g - \frac{\mu}{\nu}H + \frac{\mu^2}{2\nu^2}f\big)(y) + \big(g - \frac{\mu}{\nu}H + \frac{\mu^2}{2\nu^2}f\big)(x)f(y) + H(x)H(y).$$

Defining $G \in C(S)$ by

$$G := g - \frac{\mu}{\nu}H + \frac{\mu^2}{2\nu^2}f, \tag{5.12}$$

we can write the preceding equation as

$$f(xy) = f(x)G(y) + G(x)f(y) + H(x)H(y), \tag{5.13}$$

which is again of the form (1.1). The independence of f, H follows from (5.10). In addition we can write (5.11) in the form

$$H(xy) = H(x)G(y) + G(x)H(y) + \big(\lambda - \frac{\mu^2}{2\nu^2}\big)[f(x)H(y) + H(x)f(y)] + \Big(\frac{2\mu}{\nu} + \eta\nu\Big)H(x)H(y). \tag{5.14}$$

We apply to (5.13) the results established for (1.1) up to this point. In particular we get the equation

$$H(xy) = H(x)G(y) + G(x)H(y) + \beta^* f(x)f(y) + \gamma^*[f(x)H(y) + H(x)f(y)] + \delta^* H(x)H(y)$$

parallel to (5.2) for corresponding constants $\beta^*, \gamma^*, \delta^* \in \mathbb{C}$. Now comparing the equation above with (5.14) we find that

$$0 = -\beta^* f(x)f(y) + \big(\lambda - \frac{\mu^2}{2\nu^2} - \gamma^*\big)[f(x)H(y) + H(x)f(y)] + \Big(\frac{2\mu}{\nu} + \eta\nu - \delta^*\Big)H(x)H(y).$$

Since f and H are independent we therefore have $\beta^* = 0$, so we are back in Case 1 for the triple (f, G, H). Thus f, G, H belong to one of the families (a),(b), or (c). Since $f, g, h \in span\{f, G, H\}$ by (5.12) and (5.10), the functions f, g, h belong to the same families. This finishes Case 2.

The converse is established by Lemma 4.10. □

Note that the description of the solution in case (c) here is simpler than the one given in [1, Theorem]. We achieved this simplification by tracking the solution closely and not splitting the function $\Psi_{A',A,\chi}$ into two terms. (In [1], the function $(A' + A^2)\chi$ was split into the terms $A'\chi$ and $A^2\chi$.) This point is illustrated in the following corollary, which generalizes [1, Theorem] and is an immediate consequence of Theorem 5.1. Clearly any group satisfies the conditions on S imposed here.

Corollary 5.2. *Let S be a semigroup that satisfies $S = S^2$ and has no prime ideals. If $f, g, h\colon S \to \mathbb{C}$ satisfy (1.1) with $f \neq 0$, then the solutions belong to the following families, where $\chi, \chi', \chi_j\colon S \to \mathbb{C}$ are multiplicative functions and $A, A'\colon S \setminus I_\chi \to \mathbb{C}$ are additive functions.*

(a) $f, g, h \in V = span\{\chi_1, \chi_2, \chi_3\}$, *namely*

$$f = \sum_{j=1}^{3} a_j\chi_j, \quad g = \sum_{j=1}^{3} b_j\chi_j, \quad h = \sum_{j=1}^{3} c_j\chi_j$$

with $a_j, b_j, c_j \in \mathbb{C}$ *satisfying* (4.9).

(b) $f, g, h \in V = span\{\chi', \chi, A\chi\}$, *namely*

$$f = a_1\chi' + a_2\chi + a_3A\chi, \quad g = b_1\chi' + b_2\chi + b_3A\chi,$$
$$h = c_1\chi' + c_2\chi + c_3A\chi$$

with $a_j, b_j, c_j \in \mathbb{C}$ *satisfying* (4.11).

(c) $f, g, h \in V = span\{\chi, A\chi, (A' + \frac{1}{2}A^2)\chi\}$, *namely*

$$f = a_1\chi + a_2A\chi + a_3(A' + \frac{1}{2}A^2)\chi,$$
$$g = b_1\chi + b_2A\chi + b_3(A' + \frac{1}{2}A^2)\chi,$$
$$h = c_1\chi + c_2A\chi + c_3(A' + \frac{1}{2}A^2)\chi,$$

with $a_j, b_j, c_j \in \mathbb{C}$ *satisfying* (4.13).

Conversely, the functions in each family satisfy (1.1).

If in addition S *is a topological semigroup and* $f, g, h \in C(S)$, *then we can choose a basis* B *for* V *and coefficients* a_j, b_j, c_j *in each part above so that the coefficients are equal to* 0 *for each term not appearing in* B, *and the functions appearing in* B *belong to* $C(S)$.

6. Examples

Since the case of groups and the case of semigroups generated by their squares were handled in [1] and [2] respectively, we use semigroups which are not groups and not generated by their squares.

We start with two examples applying Corollary 3.2.

Example 6.1. Let $S = (\mathbb{N}, +)$ and suppose $f, g\colon S \to \mathbb{C}$ satisfy (1.2). There exist nonzero solutions in cases (a) and (b) of Corollary 3.2, since $S \setminus S^2 = \{1\}$ is nonempty. The exponentials on S have the form $\chi(n) = b^n$ for some $b \in \mathbb{C}^*$. Since S has no prime ideals, $\Phi_{A,\chi,\rho}$ reduces to simply $A\chi$ where

$A\colon S \to \mathbb{C}$ is additive. Such additive functions have the form $A(n) = an$ for some $a \in \mathbb{C}$. The solutions in cases (c)–(f) of Corollary 3.2 are obtained by substituting these forms into the formulas given there.

For the next example let P denote the set of primes, and for each $p \in P$ define $C_p\colon \mathbb{N} \to \mathbb{N}_0$ by

$C_p(x) :=$ the number of copies of p occurring in the prime factorization of x.

Then C_p is an additive function on the monoid $S = (\mathbb{N}, \cdot)$. For each $x \in S$ let P_x denote the set of prime factors of x, so $x = \prod_{p \in P_x} p^{C_p(x)}$.

Example 6.2. Let $S = (\mathbb{N}, \cdot)$ and suppose $f, g\colon S \to \mathbb{C}$ satisfy (1.2). Exponentials $\chi\colon S \to \mathbb{C}$ have the form $\chi(x) = \prod_{p \in P_x} \chi(p)^{C_p(x)}$ for all $x \in S$. The empty product is understood to be 1, so $\chi(1) = 1$. The prime ideals of S are of the form $\cup_{p \in Q}(p\mathbb{N})$ for nonempty proper subsets $Q \subset P$. For a given exponential χ, the additive functions $A\colon S \setminus I_\chi \to \mathbb{C}$ have the form $A(x) = \sum_{p \in P_x \setminus I_\chi} A(p)C_p(x)$ for all $x \in S \setminus I_\chi$. The empty sum is defined to be 0, so $A(1) = 0$. The set $P_\chi^{(1+)}$ is empty, and the set $P_\chi^{(1)} = I_\chi \setminus I_\chi^2$ consists of the primes $p \in I_\chi$ and their products with elements of $S \setminus I_\chi$. Condition (II) of Proposition 2.1 states that $\rho(x) = \rho(p)\chi(w)$ for $x = pw$ with $p \in P_\chi^{(1)}$ and $w \in S \setminus I_\chi$. Here the value of $\rho(p)$ for each $p \in P \cap I_\chi$ is arbitrary, and the values of ρ at all other points of $P_\chi^{(1)}$ are determined by condition (II).

Solutions of (1.2) are obtained by substituting these forms into the formulas in cases (c)–(f) of Corollary 3.2. Since S is a monoid, case (a) yields only $f = g = 0$ and case (b) is vacuous.

Now we turn to examples illustrating our results about (1.1).

Example 6.3. Let $S = (-1, 0) \cup (0, 1)$ under multiplication and the usual topology. Clearly $S = S^2$ and S has no prime ideals, so we can apply Corollary 5.2 to get the solutions of (1.1) with $f \neq 0$. The continuous exponentials on S have one of the three forms

$$\chi = 1, \quad \chi(x) = |x|^\alpha, \quad \text{or} \quad \chi(x) = |x|^\alpha sgn(x),$$

where $\alpha \in \mathbb{C}$ has positive real part. The continuous additive functions on S are of the form $A(x) = c \log |x|$ for some $c \in \mathbb{C}$.

The next example has two prime ideals, $I_1 = \{0\}$ and $I_2 = (-1, 1)$, both of which satisfy $I = I^2$. We choose a topological version that eliminates I_2 for convenience. (The exponential χ with $I_\chi = I_2$ is defined by $\chi(1) = \chi(-1) = 1$ and $\chi(x) = 0$ for $-1 < x < 1$.)

Example 6.4. Let $S = [-1, 1]$ under multiplication and the usual topology. Then S is t-compatible, so we can apply Theorem 5.1 to get the continuous solutions of (1.1) with $f \neq 0$. The continuous exponentials on S have one of the three forms

$$\chi = 1, \quad \chi(x) = \begin{cases} |x|^\alpha & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{or} \quad \chi(x) = \begin{cases} |x|^\alpha sgn(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\alpha \in \mathbb{C}$ has positive real part. The continuous additive functions on $S \setminus \{0\}$ are of the form $A(x) = c \log |x|$ for some $c \in \mathbb{C}$. The only additive function on S is the zero function, so $\Phi_{A,1} = \Psi_{A',A,1} = 0$. Thus solution families (b) and (c) arise (non-trivially) only for $\chi \neq 1$.

The final example illustrates some of the complexity obstructing attempts to solve (1.1) on a general semigroup. We return to $S = (\mathbb{N}, \cdot)$, which satisfies $S = S^2$ since it is a monoid. As we saw in Example 6.2 we can get the solutions of (1.2) on S from Corollary 3.2. But S is not a compatible semigroup, since $I = p\mathbb{N}$ is a prime ideal for any prime p and $pw \in I \setminus I^2$ for every $w \in S \setminus I$, thus condition (4.1) of compatibility fails.

The following example exhibits solutions of (1.1) that are not of the forms in Theorem 5.1.

Example 6.5. Let $S = (\mathbb{N}, \cdot)$, let $p \neq q$ be primes, let $I = p\mathbb{N} \cup q\mathbb{N}$, and let $g \colon S \to \mathbb{C}$ be the exponential

$$g(x) = \begin{cases} 1 & \text{for } x \in S \setminus I, \\ 0 & \text{for } x \in I, \end{cases}$$

with null ideal I. Let $h \colon S \to \mathbb{C}$ be defined by

$$h(x) = \begin{cases} 0 & \text{for } x \in S \setminus I, \\ \rho(s) & \text{for } x = sw \in I \setminus I^2, \text{ with } s \in \{p, q\}, \ w \in S \setminus I, \\ 0 & \text{for } x \in I^2, \end{cases}$$

where $\rho \colon \{p, q\} \to \mathbb{C}$ is not the zero function. Define $f \colon S \to \mathbb{C}$ by

$$f(x) = \begin{cases} A'(x) & \text{for } x \in S \setminus I, \\ \rho'(s) & \text{for } x = sw \in I \setminus I^2 \text{ with } s \in \{p, q\}, \ w \in S \setminus I, \\ \rho(s)\rho(t) & \text{for } x = stw \in I^2 \setminus I^3 \text{ with } s, t \in \{p, q\}, \ w \in S \setminus I, \\ 0 & \text{for } x \in I^3, \end{cases}$$

for some additive $A' \colon S \setminus I_\chi \to \mathbb{C}$ and a nonzero function $\rho' \colon \{p, q\} \to \mathbb{C}$.

It can be checked that f, g, h satisfy (1.1), with $f \neq 0$ and $h = \Phi_{0,g,\rho} \neq 0$. Clearly neither f nor h has a form seen in Theorem 5.1.

Things can get more complicated than the last example. For other semigroups (and exponentials χ defined on them), all three pieces of the nullspace partition $I_\chi = I_\chi^2 \cup P_\chi^{(1+)} \cup P_\chi^{(1)}$ are nonempty, and the interactions under multiplication of elements from different pieces can complicate the picture further. The proliferation of cases created by this situation makes the problem of solving (1.1) on a general semigroup rather unwieldy.

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