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A VARIANT OF D'ALEMBERT'S FUNCTIONAL EQUATION ON SEMIGROUPS WITH ENDOMORPHISMS

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Abstract. Let S be a semigroup, and let $\varphi, \psi\colon S \to S$ be two endomorphisms (which are not necessarily involutive). Our main goal in this paper is to solve the following generalized variant of d'Alembert's functional equation

$$f(x\varphi(y)) + f(\psi(y)x) = 2f(x)f(y), \quad x, y \in S,$$

where $f\colon S \to \mathbb{C}$ is the unknown function by expressing its solutions in terms of multiplicative functions. Some consequences of this result are presented.

1. Set up and notation

To formulate our results we recall the following notations and notions that will be used throughout the paper. Let S be a semigroup, i.e., a set equipped with an associative operation. Let $\varphi, \psi\colon S \to S$ be two endomorphisms, i.e., $\varphi(xy) = \varphi(x)\varphi(y)$ and $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in S$. The function $\chi\colon S \to \mathbb{C}$ is said to be multiplicative, if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$, and furthemore if $\chi(x) \neq 0$ for all $x \in S$, then χ is said a character. Also if $\chi \circ \varphi = \chi$, then χ is said φ-even. The function $\chi\colon S \to \mathbb{C}$ is said to be central if $\chi(xy) = \chi(yx)$ for all $x, y \in S$. If S is a topological semigroup, then we let $C(S)$ denote the algebra of continuous functions from S into $\mathbb{C}$.

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2. Introduction

The functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \quad x,y\in\mathbb{R}, \tag{2.1}$$

is known in the literature as the d'Alembert functional equation. It has a long history going back to d'Alembert ([5]). As the name suggests this functional equation was introduced by d'Alembert in connection with the composition of forces and plays a central role in determining the sum of two vectors in Euclidean and non-Euclidean geometries. The continuous solutions of (2.1) were determined by Cauchy in 1821 (see [3]). The equation (2.1) has been extended to abelian groups: You just replace the domain of definition $\mathbb{R}$ by an abelian group $(G,+)$. It was resolved in this setting. The functional equation (2.1) was generalized to a semigroup S by the equation

$$f(xy)+f(\tau(y)x)=2f(x)f(y), \quad x,y\in S, \tag{2.2}$$

where τ is an involutive automorphism (i.e., $\tau^2=id$), which was introduced and solved by Stetkær in [14]. In [7], Fadli et al. have solved Eq (2.2) in the case where τ is an arbitrary endomorphism of S. Also the functional equation (2.1) was generalized to a semigroup S with two involutive automorphisms by the following equation

$$f(x\sigma(y))+f(\tau(y)x)=2f(x)f(y), \quad x,y\in S, \tag{2.3}$$

which was solved in [4] by Chahbi et al. (σ,τ are two involutive automorphisms). This last equation generalizes Eq (2.2).

Some information, applications and numerous references concerning (2.1)–(2.3) and their further generalizations can be found e.g. in [1, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Our main objective in the present paper is to solve the following functional equation

$$f(x\varphi(y))+f(\psi(y)x)=2f(x)f(y), \quad x,y\in S, \tag{2.4}$$

where φ,ψ are two endomorphisms of S. This equation is a natural generalization of Eqs (2.1)–(2.3). By elementary methods we find all solutions of (2.4) on semigroups in terms of multiplicative functions.

If $\varphi = \psi = id$, the equation (2.4) is the following symmetrized multiplicative Cauchy equation

$$f(xy) + f(yx) = 2f(x)f(y), \quad x, y \in S,$$

which was introduced and solved by Stetkær in [12].

In section 3, we study also some important special cases of (2.4), exactly the case when φ and/or ψ are surjective, the case when φ and ψ are both surjective and the case when $\varphi = \psi$.

Our main contribution in this paper is to solve the equation (2.4) without the conditions $\varphi^2 = id$ and $\psi^2 = id$.

Finally, we note that the sine addition law on semigroups given in [6, 13] is a key ingredient of the proof of our main result (Theorem 3.3).

3. Main result

In this section, we solve the functional equation (2.4) by expressing its solutions in terms of multiplicative functions. The following lemmas will be used in the proof of Theorem 3.3.

LEMMA 3.1. *Let $f\colon S \to \mathbb{C}$ be a function such that*

$$f(x\varphi(y)) = f(x)f(y), \quad \textit{for all } x, y \in S. \tag{3.1}$$

Then f is a multiplicative function.

PROOF. Let $x, y, z \in S$ and let $f\colon S \to \mathbb{C}$ be a function satisfying (3.1). Then

$$f(z)f(xy) = f(z\varphi(xy)) = f(z\varphi(x)\varphi(y)) = f(z\varphi(x))f(y) = f(z)f(x)f(y).$$

Hence f is a multiplicative function. □

LEMMA 3.2. *Let $f\colon S \to \mathbb{C}$ be a function. For all $x \in S$, define the function $h_x\colon S \to \mathbb{C}$ by*

$$h_x(y) := f(x\varphi(y)) - f(x)f(y), \quad y \in S.$$

If f is a solution of (2.4), then the pair (h_x, f) satisfies the sine addition law for all $x \in S$.

Proof. Let $x, y, z \in S$ and let $f\colon S \to \mathbb{C}$ be a solution of (2.4). Making the substitutions (x, yz), $(x\varphi(y), z)$ and $(\psi(z)x, y)$ in (2.4), we get respectively

$$f(x\varphi(yz)) + f(\psi(yz)x) = 2f(x)f(yz), \tag{3.2}$$

$$f(x\varphi(y)\varphi(z)) + f(\psi(z)x\varphi(y)) = 2f(x\varphi(y))f(z), \tag{3.3}$$

$$f(\psi(z)x\varphi(y)) + f(\psi(y)\psi(z)x) = 2f(\psi(z)x)f(y). \tag{3.4}$$

Subtracting (3.4) from the sum of (3.2) and (3.3), we get that

$$f(x\varphi(yz)) = f(x)f(yz) + f(x\varphi(y))f(z) - f(\psi(z)x)f(y). \tag{3.5}$$

From (2.4), we see that

$$f(\psi(z)x) = 2f(x)f(z) - f(x\varphi(z)).$$

Then (3.5) becomes after a reduction

$$\begin{aligned} f(x\varphi(yz)) - f(x)f(yz) = {} & [f(x\varphi(y)) - f(x)f(y)]f(z) \\ & + [f(x\varphi(z)) - f(x)f(z)]f(y). \end{aligned}$$

So

$$h_x(yz) = h_x(y)f(z) + h_x(z)f(y), \quad y, z \in S,$$

i.e., (h_x, f) satisfies the sine addition law. □

Now we present our main result.

Theorem 3.3. *The solutions $f\colon S \to \mathbb{C}$ of (2.4) are the following:*

(i) *There exists a non-zero multiplicative function $\chi\colon S \to \mathbb{C}$ satisfying ($\chi \circ \varphi = \chi$ and $\chi \circ \psi = 0$) or ($\chi \circ \psi = \chi$ and $\chi \circ \varphi = 0$) such that $f = \frac{1}{2}\chi$.*

(ii) *There exists a multiplicative function $\chi\colon S \to \mathbb{C}$ satisfying*

$$\chi \circ \varphi + \chi \circ \psi = \chi \circ \varphi^2 + \chi \circ \varphi \circ \psi = \chi \circ \psi^2 + \chi \circ \psi \circ \varphi$$

such that

$$f = \frac{\chi \circ \varphi + \chi \circ \psi}{2}.$$

Furthermore, if S is a topological semigroup and $f \in C(S)$ then $\chi \circ \varphi, \chi \circ \psi \in C(S)$.

PROOF. It is easy to check that any function of the form (i) or (ii) stated in Theorem 3.3 is a solution of (2.4). Conversely assume that the function $f\colon S \to \mathbb{C}$ is a solution of (2.4). From Lemma 3.2, the pair (h_x, f), where

$$h_x(y) = f(x\varphi(y)) - f(x)f(y), \quad y \in S,$$

satisfies the sine addition law for all $x \in S$.

Case 1: Suppose that $h_x = 0$ for all $x \in S$. By the definition of h_x we have

$$f(x\varphi(y)) = f(x)f(y) \quad \text{for all } x, y \in S.$$

According to Lemma 3.1, we see that f is multiplicative. Then (2.4) gives

$$2f = f \circ \varphi + f \circ \psi.$$

From the theory of multiplicative functions, we obtain $f = f \circ \varphi = f \circ \psi$. Hence $f = (f \circ \varphi + f \circ \psi)/2$ and

$$f \circ \varphi + f \circ \psi = f \circ \varphi^2 + f \circ \varphi \circ \psi = f \circ \psi^2 + f \circ \psi \circ \varphi.$$

So we are in case (ii) of Theorem 3.3.

Case 2: Suppose that $h_x \neq 0$ for some $x \in S$. From the known solution of the sine addition law (see for instance [6] or [13, Theorem 4.1]), there exist two multiplicative functions $\chi_1, \chi_2\colon S \to \mathbb{C}$ such that

$$f = \frac{\chi_1 + \chi_2}{2}.$$

If $\chi_1 = \chi_2$, we get that $f = \chi_1 = \chi_2$, which we are in the first case studied.

Suppose that $\chi_1 \neq \chi_2$. Substituting f in (2.4), we find after a reduction that

$$\chi_1(x)[\chi_1 \circ \varphi(y) + \chi_1 \circ \psi(y) - \chi_1(y) - \chi_2(y)] + \chi_2(x)[\chi_2 \circ \varphi(y) + \chi_2 \circ \psi(y) - \chi_1(y) - \chi_2(y)] = 0,$$

for all $x, y \in S$. Since $\chi_1 \neq \chi_2$ we get from the theory of multiplicative functions (see for instance [13, Theorem 3.18]) that both terms are 0, so

$$\begin{cases} \chi_1(x)[\chi_1 \circ \varphi(y) + \chi_1 \circ \psi(y) - \chi_1(y) - \chi_2(y)] = 0, \\ \chi_2(x)[\chi_2 \circ \varphi(y) + \chi_2 \circ \psi(y) - \chi_1(y) - \chi_2(y)] = 0, \end{cases} \tag{3.6}$$

for all $x, y \in S$. Since $\chi_1 \neq \chi_2$ at least one of χ_1 and χ_2 is not zero.

Subcase 2.1: Suppose that $\chi_2 = 0$. Then $\chi_1 \neq 0$. From (3.6), we infer that $\chi_1 = \chi_1 \circ \varphi + \chi_1 \circ \psi$. Therefore $f = \frac{1}{2}\chi_1$ and ($\chi_1 = \chi_1 \circ \varphi$ and $\chi_1 \circ \psi = 0$) or ($\chi_1 = \chi_1 \circ \psi$ and $\chi_1 \circ \varphi = 0$). With $\chi := \chi_1$ we arrive at solution in case (i) of Theorem 3.3. The same result we obtain for $\chi_1 = 0$ and $\chi_2 \neq 0$ with $\chi := \chi_2$.

Subcase 2.2: Suppose that $\chi_1 \neq 0$ and $\chi_2 \neq 0$. From (3.6), we have

$$\chi_1 + \chi_2 = \chi_1 \circ \varphi + \chi_1 \circ \psi = \chi_2 \circ \varphi + \chi_2 \circ \psi.$$

Then necessarily $\chi_1 \circ \varphi \neq 0$ and $\chi_1 \circ \psi \neq 0$. Now substituting f into (2.4) again, but in this case by $(\chi_1 \circ \varphi + \chi_1 \circ \psi)/2$, we obtain, from the theory of multiplicative functions, after a reduction

$$\chi_1 \circ \varphi + \chi_1 \circ \psi = \chi_1 \circ \varphi^2 + \chi_1 \circ \varphi \circ \psi = \chi_1 \circ \psi^2 + \chi_1 \circ \psi \circ \varphi.$$

So we are in the solution stated in (ii) of Theorem 3.3 with $\chi = \chi_1$.

In view of these cases, we have reached the end of the proof.

The continuity statement follows from [13, Theorem 3.18(d)]. □

As immediate consequences of Theorem 3.3, we have the following corollaries. The first generalizes the result (Theorem 3.1) studied in [7].

Corollary 3.4. *Suppose that φ and/or ψ are surjective. The solutions $f\colon S \to \mathbb{C}$ of (2.4) are the following:*

(i) *There exists a non-zero multiplicative function $\chi\colon S \to \mathbb{C}$ such that $f = \frac{1}{2}\chi$ with ($\chi \circ \varphi = \chi$ and $\chi \circ \psi = 0$) or ($\chi \circ \psi = \chi$ and $\chi \circ \varphi = 0$).*

(ii) *There exists a multiplicative function $\chi\colon S \to \mathbb{C}$ such that*

$$f = \frac{\chi \circ \varphi + \chi \circ \psi}{2}$$

with the conditions:

(1) $\chi \circ \varphi + \chi \circ \psi = \chi \circ \varphi^2 + \chi \circ \varphi \circ \psi$,

(2) $\chi \circ \varphi \circ \psi = \chi \circ \psi \circ \varphi$, *and*

(3) $\chi \circ \varphi^2 = \chi \circ \psi^2$.

Furthermore, if S is a topological semigroup and $f \in C(S)$, then $\chi \circ \varphi, \chi \circ \psi \in C(S)$.

Proof. Let $f\colon S \to \mathbb{C}$ be a solution of (2.4). From Theorem 3.3, we have the two following possibilities:

(1) There exists a non-zero multiplicative function $\chi\colon S \to \mathbb{C}$ such that $f = \frac{1}{2}\chi$ with ($\chi \circ \varphi = \chi$ and $\chi \circ \psi = 0$) or ($\chi \circ \psi = \chi$ and $\chi \circ \varphi = 0$), which is also the first case of Corollary 3.4.

(2) There exists a multiplicative function $\chi\colon S \to \mathbb{C}$ such that $f = (\chi \circ \varphi + \chi \circ \psi)/2$ with the condition

$$\chi \circ \varphi + \chi \circ \psi = \chi \circ \varphi^2 + \chi \circ \varphi \circ \psi = \chi \circ \psi^2 + \chi \circ \psi \circ \varphi. \tag{3.7}$$

Case: 1 If $\chi \circ \varphi = \chi \circ \psi$. Then $\chi \circ \varphi^2 = \chi \circ \psi^2 = \chi \circ \varphi \circ \psi = \chi \circ \psi \circ \varphi$.

Case: 2 If $\chi \circ \varphi \neq \chi \circ \psi$. Then if $\chi \circ \varphi^2 = \chi \circ \psi \circ \varphi$ and $\chi \circ \psi^2 = \chi \circ \varphi \circ \psi$, we get that $\chi \circ \varphi = \chi \circ \psi$ because φ and/or ψ are surjective, which is not possible with $\chi \circ \varphi \neq \chi \circ \psi$. So from (3.7), we have $\chi \circ \varphi \circ \psi = \chi \circ \psi \circ \varphi$ and $\chi \circ \varphi^2 = \chi \circ \psi^2$. The other direction of the proof is trivial to verify.

The continuity statement follows from [13, Theorem 3.18(d)]. □

The following corollary generalizes the result studied in [4].

Corollary 3.5. *Suppose that φ and ψ are both surjective. Then the solutions $f\colon S \to \mathbb{C}$ of (2.4) are the functions of the form*

$$f = \frac{\chi \circ \varphi + \chi \circ \psi}{2},$$

where $\chi\colon S \to \mathbb{C}$ is a multiplicative function such that
(1) $\chi \circ \varphi + \chi \circ \psi = \chi \circ \varphi^2 + \chi \circ \varphi \circ \psi$,
(2) $\chi \circ \varphi \circ \psi = \chi \circ \psi \circ \varphi$, *and*
(3) $\chi \circ \varphi^2 = \chi \circ \psi^2$.
Furthermore, if S is a topological semigroup and $f \in C(S)$, then $\chi \circ \varphi, \chi \circ \psi \in C(S)$.

Proof. The proof follows from the fact that $(f \circ \varphi = 0 \Rightarrow f = 0)$ and $(f \circ \psi = 0 \Rightarrow f = 0)$ when φ and ψ are both surjective. The option (i) of Corollary 3.4 does not occur in this case. □

Corollary 3.6. *Suppose that φ is surjective. The solutions $f\colon S \to \mathbb{C}$ of the functional equation*

$$f(x\varphi(y)) + f(\varphi(y)x) = 2f(x)f(y), \quad x, y \in S,$$

are the φ-even multiplicative functions.

Proof. The proof follows on putting $\varphi = \psi$ in Corollary 3.5 and the fact that $\chi \circ \varphi = \chi \circ \varphi^2$ implies that $\chi = \chi \circ \varphi$ when φ is surjective. □

The following corollary gives the central solutions of the functional equation

$$f(x\varphi(y)) + f(x\psi(y)) = 2f(x)f(y), \quad x, y \in S, \tag{3.8}$$

which is a natural generalization of d'Alembert functional equation (2.1). With $\varphi = id$ and ψ is a continuous anti-endomorphism (i.e., $\psi(xy) = \psi(y)\psi(x)$ for all $x, y \in S$), the equation (3.8) was studied and solved on topological monoid (i.e., a semigroup with an identity element) in [2].

COROLLARY 3.7. *The central solutions $f\colon S \to \mathbb{C}$ of (3.8) are the following:*

(1) *There exists a non-zero multiplicative function $\chi\colon S \to \mathbb{C}$ satisfying ($\chi \circ \varphi = \chi$ and $\chi \circ \psi = 0$) or ($\chi \circ \psi = \chi$ and $\chi \circ \varphi = 0$) such that $f = \frac{1}{2}\chi$.*

(2) *There exists a multiplicative function $\chi\colon S \to \mathbb{C}$ satisfying*

$$\chi \circ \varphi + \chi \circ \psi = \chi \circ \varphi^2 + \chi \circ \varphi \circ \psi = \chi \circ \psi^2 + \chi \circ \psi \circ \varphi$$

such that

$$f = \frac{\chi \circ \varphi + \chi \circ \psi}{2}.$$

Furthermore, if S is a topological semigroup and $f \in C(S)$ then $\chi \circ \varphi, \chi \circ \psi \in C(S)$.

PROOF. It suffices to observe that if f is a central function then the equations (2.4) and (3.8) are equivalent. □

4. Some examples

EXAMPLE 4.1. Let $S = H_3$ be the Heisenberg group (under matrix multiplication) defined by

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

and let us consider the following endomorphisms on S

$$\varphi\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\psi\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x-y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that φ, ψ are not involutive automorphisms and are not surjective. We have $\varphi^2 = \varphi$, $\psi^2 = \psi$, $\varphi \circ \psi = \psi$ and $\psi \circ \varphi = \varphi$. In this example, we determine the corresponding non-zero continuous solutions of (2.4). According to [13, Example 3.14], the continuous non-zero multiplicative functions on S are:

$$\chi_\lambda : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto e^{\lambda_1 x + \lambda_2 y},$$

where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$. The function χ_λ satisfies the condition

$$\chi \circ \varphi + \chi \circ \psi = \chi \circ \varphi^2 + \chi \circ \varphi \circ \psi = \chi \circ \psi^2 + \chi \circ \psi \circ \varphi.$$

Since χ_λ is a character (because H_3 is a group), we have $\chi_\lambda \circ \varphi \neq 0$ and $\chi_\lambda \circ \psi \neq 0$.

So the option (i) of Theorem 3.3 does not occur here. We have also

$$\chi_\lambda \circ \varphi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = e^{\lambda_1 (x+y)}$$

and

$$\chi_\lambda \circ \psi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = e^{\lambda_1 (x-y)}.$$

Consequently, from Theorem 3.3, the corresponding non-zero continuous solutions of (2.4) are the functions

$$f : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto e^{\gamma x} \cosh(\gamma y),$$

where $\cosh(z) := \dfrac{e^z + e^{-z}}{2}$ for all $z \in \mathbb{C}$ and γ ranges over $\mathbb{C}$.

EXAMPLE 4.2. Let $S = [0,1]^2$ be the square of the closed unit interval under componentwise multiplication, so $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2)$. Let $\varphi, \psi \colon S \to S$ be two endomorphisms defined by

$$\varphi(x_1, x_2) = (x_1, 0) \quad \text{and} \quad \psi(x_1, x_2) = (0, x_2) \quad \text{for all } (x_1, x_2) \in S.$$

Note that $\varphi^2 = \varphi$, $\psi^2 = \psi$, $\varphi \circ \psi = \psi \circ \varphi = (0,0)$. We determine here the corresponding continuous solutions of (2.4). We write $R(\alpha)$ for the real part of the complex number α. For convenience we define here $0^\alpha := 0$ when $R(\alpha) > 0$.

The continuous non-zero multiplicative functions on S are of the following four types (see for instance [6, Example 5.3]).

(i) There exists $(\alpha, \beta) \in \mathbb{C}^2$ with $R(\alpha) > 0$, $R(\beta) > 0$ such that

$$\chi_{\alpha,\beta}(x_1, x_2) = x_1^\alpha x_2^\beta \quad \text{for } (x_1, x_2) \in S.$$

(ii) There exists $\alpha \in \mathbb{C}$ with $R(\alpha) > 0$ such that

$$\chi_{\alpha,0}(x_1, x_2) = x_1^\alpha \quad \text{for } (x_1, x_2) \in S.$$

(iii) There exists $\beta \in \mathbb{C}$ with $R(\beta) > 0$ such that

$$\chi_{0,\beta}(x_1, x_2) = x_2^\beta \quad \text{for } (x_1, x_2) \in S.$$

(iv) $\chi_{0,0}(x_1, x_2) = 1$ for $(x_1, x_2) \in S$.

Let $\chi \neq 0$ be an arbitrary continuous multiplicative function on S. Then

(1) If χ has the form stated in case (i), we have $\chi \circ \varphi = \chi \circ \psi = 0$.
(2) If χ has the form stated in case (ii), we have $\chi \circ \varphi = \chi$ and $\chi \circ \psi = 0$.
(3) If χ has the form stated in case (iii), we have $\chi \circ \varphi = 0$ and $\chi \circ \psi = \chi$.
(4) If χ has the form stated in case (iv), we have $\chi \circ \varphi = \chi \circ \psi = 1$.

In conclusion, using Theorem 3.3 we find that the solutions $f \in C(S)\setminus\{0\}$ of (2.4), which is

$$f(x_1y_1,0)+f(0,x_2y_2)=2f(x_1,x_2)f(y_1,y_2), \quad x_1,x_2,y_1,y_2\in[0,1],$$

are the following:

(1) There exists $\alpha\in\mathbb{C}$ with $R(\alpha)>0$ such that

$$f(x_1,x_2)=\frac{1}{2}\chi_{\alpha,0}(x_1,x_2)=\frac{1}{2}x_1^{\alpha} \quad \text{for } (x_1,x_2)\in S.$$

(2) There exists $\beta\in\mathbb{C}$ with $R(\beta)>0$ such that

$$f(x_1,x_2)=\frac{1}{2}\chi_{0,\beta}(x_1,x_2)=\frac{1}{2}x_2^{\beta} \quad \text{for } (x_1,x_2)\in S.$$

(3) $f(x_1,x_2)=\chi_{0,0}(x_1,x_2)=1 \quad \text{for } (x_1,x_2)\in S.$

Note that the only non-zero continuous solution of case (ii) in Theorem 3.3 is $f=1$.

Example 4.3. Let $S=(\mathbb{R},+)$, let $\alpha,\beta\in\mathbb{R}\setminus\{0\}$ be two fixed elements and let $\varphi(x)=\alpha x$, $\psi(x)=\beta x$ for all $x\in\mathbb{R}$. Then the functional equation (2.4) is written as follows:

$$f(x+\alpha y)+f(x+\beta y)=2f(x)f(y), \quad x,y\in\mathbb{R}. \tag{4.1}$$

We note that the equation (4.1) is d'Alembert's equation (2.1) when $\alpha=1$ and $\beta=-1$. When $\alpha=1$ and $\beta\in\mathbb{R}\setminus\{0,1\}$ the equation (4.1) was studied by Fadli et al. in [7]. We are interested to determine the solutions of this equation when $\alpha,\beta\in\mathbb{R}\setminus\{-1,0,1\}$. For this we apply Corallary 3.5 to equation (4.1): there exists a multiplicative function $\chi\colon\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=\frac{\chi(\alpha x)+\chi(\beta x)}{2}, \quad x\in\mathbb{R},$$

with the conditions

$$\chi(\alpha x)+\chi(\beta x)=\chi(\alpha^2x)+\chi(\alpha\beta x), \tag{4.2}$$

$$\chi(\alpha^2x)=\chi(\beta^2x). \tag{4.3}$$

Suppose that $\chi\neq0$. Then χ is a character because, $(\mathbb{R},+)$ is a group. So the identity (4.3) is equivalent to $\chi((\alpha^2-\beta^2)x)=1$ for all $x\in\mathbb{R}$.

If $\beta \neq \pm\alpha$, we obtain $\chi(x) = 1$ for all $x \in \mathbb{R}$, because the map $x \mapsto (\alpha^2 - \beta^2)x$ is surjective on $\mathbb{R}$ when $\beta \neq \pm\alpha$. Hence $f = 1$.

If $\beta = \alpha$, then $f(x) = \chi(\alpha x)$ for all $x \in \mathbb{R}$. From (4.2), $\chi(\alpha x) = \chi(\alpha^2 x)$, that implies $\chi(\alpha(\alpha - 1)x) = 1$ for all $x \in \mathbb{R}$. Since $\alpha(\alpha - 1) \neq 0$, then $\chi = 1$. Hence $f = 1$.

If $\beta = -\alpha$, then (4.2) becomes $\chi(\alpha x) - \chi(-\alpha^2 x) = \chi(\alpha^2 x) - \chi(-\alpha x)$ for all $x \in \mathbb{R}$. We have the following equivalences:

$$\begin{aligned} &\chi(\alpha x) - \chi(-\alpha^2 x) = \chi(\alpha^2 x) - \chi(-\alpha x) \\ &\quad \iff \quad [\chi(\alpha x + \alpha^2 x) - 1]\chi(-\alpha^2 x) = [\chi(\alpha x + \alpha^2 x) - 1]\chi(-\alpha x) \\ &\quad \iff \quad [\chi(\alpha x + \alpha^2 x) - 1][\chi(-\alpha^2 x) - \chi(-\alpha x)] = 0 \\ &\quad \iff \quad [\chi(\alpha x + \alpha^2 x) - 1][\chi(-\alpha^2 x + \alpha x) - 1] = 0. \end{aligned}$$

So $\chi((\alpha + \alpha^2)x) = 1$ or $\chi((\alpha - \alpha^2)x) = 1$. Since $\alpha + \alpha^2 \neq 0$ and $\alpha - \alpha^2 \neq 0$ (because $\alpha \neq 0$ and $\alpha \neq \pm 1$), we obtain $\chi = 1$. So $f = 1$.

Conclusion: If $\alpha, \beta \in \mathbb{R} \setminus \{-1, 0, 1\}$, the solutions of (4.1) are $f = 0$ and $f = 1$.

Example 4.4. Let G be the $(ax + b)-$group defined by

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a > 0, \quad b \in \mathbb{R} \right\}$$

and let us consider the following endomorphisms on G

$$\varphi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\psi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that φ, ψ are not involutive. In this example, we give the corresponding continuous central solutions of (3.8).

According to [13, Example 3.13], the continuous non-zero multiplicative functions on G have the form

$$\chi_\lambda : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto a^\lambda,$$

where $\lambda \in \mathbb{C}$.

We observe that $\varphi^2 = \psi^2 = \varphi$ and $\varphi \circ \psi = \psi \circ \varphi = \psi$. Then the non-zero continuous multiplicative function χ defined above on G satisfies
$\chi \circ \varphi + \chi \circ \psi = \chi \circ \varphi^2 + \chi \circ \varphi \circ \psi = \chi \circ \psi^2 + \chi \circ \psi \circ \varphi$.

As a conclusion, from Corollary 3.7 we obtain that the non-zero continuous central solutions $f\colon G \to \mathbb{C}$ of (3.8) are:

$$f : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \frac{a^{\lambda} + a^{-\lambda}}{2},$$

where $\lambda \in \mathbb{C}$.

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