

SUMMING A FAMILY OF GENERALIZED PELL NUMBERS

HELMUT PRODINGER 

Abstract. A new family of generalized Pell numbers was recently introduced and studied by Bród ([2]). These numbers possess, as Fibonacci numbers, a Binet formula. Using this, partial sums of arbitrary powers of generalized Pell numbers can be summed explicitly. For this, as a first step, a power P_n^l is expressed as a linear combination of P_{mn} . The summation of such expressions is then manageable using generating functions. Since the new family contains a parameter $R = 2^r$, the relevant manipulations are quite involved, and computer algebra produced huge expressions that were not trivial to handle at times.

1. Introduction

The Pell numbers P_n are defined via the recursion $P_n = 2P_{n-1} + P_{n-2}$ and $P_0 = 0$, $P_1 = 1$; they are sequence A000129 in [4], they possess a Binet formula

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}$$

and a generating function

$$\sum_{n \geq 0} P_n z^n = \frac{z}{1 - 2z - z^2}.$$

Received: 28.10.2020. Accepted: 19.11.2020. Published online: 14.12.2020.

(2020) Mathematics Subject Classification: 11B49, 05A15.

Key words and phrases: Pell numbers, Binet formula, generating functions.

©2020 The Author(s).

This is an Open Access article distributed under the terms of the Creative Commons Attribution License CC BY (<http://creativecommons.org/licenses/by/4.0/>).

A non-complete list of references about Pell numbers and generalizations is here: [1, 3, 10, 6]. The search engine MathSciNet lists almost 1000 entries for the keyword ‘Pell numbers’.

Bród ([2]) recently introduced r -Pell numbers, via

$$P(r, n) = C_1 r_1^n + C_2 r_2^n,$$

with

$$r_{1,2} = \frac{2^r \pm \sqrt{4^r + 2^{r+1}}}{2}, \quad C_{1,2} = 1 \pm \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}.$$

The classical Pell numbers are then given as $P_{n+2} = P(1, n)$. First, we write $R = 2^r$, and $W = \sqrt{R^2 + 2R}$. Then we have

$$C_1 r_1^n + C_2 r_2^n = \frac{2}{RW} (r_1^{n+2} - r_2^{n+2}),$$

and we will consider the shifted versions

$$P_R(n) := \frac{2}{RW} (\lambda^n - \mu^n), \quad \text{with} \quad \lambda = \frac{R + \sqrt{R^2 + 4R}}{2}, \quad \mu = \frac{R - \sqrt{R^2 + 4R}}{2}.$$

This version is slightly more appealing, and fits better with the classical Pell numbers: $P_2(n) = P(n) = P_n$. (For $R = 1$ we are in the world of Fibonacci numbers.) The recursion is

$$P_R(n+2) - 2RP_R(n+1) - RP_R(n) = 0,$$

and the first few numbers are

$$0, \frac{2}{R}, 2, 2R+1, 2R(R+1), \frac{R}{2} (4R^2 + 6R + 1), \frac{R^2}{2} (2R+3)(2R+1), \dots$$

We also need the associated sequence

$$Q_R(n) := \frac{2}{R} (\lambda^n + \mu^n),$$

and it is easy to see that $Q_R(n) = 2P_R(n+1) - RP_R(n)$.

The goal of the present note is to compute the sums

$$\sum_{0 \leq k \leq n} P_R(km)^\ell$$

efficiently, for positive parameters m and ℓ .

We presented such a treatment in [8] for Fibonacci numbers, and, more recently, also in [9] for balancing numbers studied by Komatsu ([5]), but the current instance is definitely the most challenging, as it contains the parameter R , and that makes computations difficult (even for a computer).

2. Generating functions

First we give the generating function for the generalized Pell numbers, which follows immediately from the Binet formula:

$$\sum_{n \geq 0} P_R(n) z^n = \frac{4z}{R(2 - 2zR - z^2R)}.$$

We have

$$\begin{aligned} 2^m P_R((n+2)m) - (R(R+1)2^{m-1}P_R(m) - R^2 2^{m-2}P_R(m-2)) P_R((n+1)m) \\ + (-1)^m R^m P_R(nm) = 0. \end{aligned}$$

For $m = 1$, this is just the ordinary recursion for generalized Pell numbers, but we need this more general version.

This formula is not too difficult to prove using the Binet formula and the help of a computer, but not so easy to find. We used Maple's package `gfun` for that. Let us rewrite this in terms of generating functions: Set

$$F_m(z) = \sum_{n \geq 0} P_R(mn) z^n,$$

then

$$\begin{aligned} 2^m \sum_{n \geq 2} P_R(nm) z^n \\ - z(R(R+1)2^{m-1}P_R(m) - R^2 2^{m-2}P_R(m-2)) \sum_{n \geq 1} P_R(nm) z^n \\ + z^2 (-1)^m R^m \sum_{n \geq 0} P_R(nm) z^n = 0 \end{aligned}$$

or

$$\begin{aligned} 2^m (F_m(z) - z P_R(m)) - z(R(R+1)2^{m-1}P_R(m) - R^2 2^{m-2}P_R(m-2)) F_m(z) \\ + z^2 (-1)^m R^m F_m(z) = 0, \end{aligned}$$

or

$$F_m(z) = \frac{2^m z P_R(m)}{2^m - z(R(R+1)2^{m-1}P_R(m) - R^2 2^{m-2}P_R(m-2)) + (-1)^m z^2 R^m}.$$

For $m = 1$, we get the first generating function again as a check.

Now let us abbreviate

$$F_m(z) = \frac{az}{b + cz + dz^2},$$

then

$$\frac{1}{1-z} F_m(z) = \frac{a}{b+c+d} \frac{1}{1-z} - \frac{a}{b+c+d} \frac{b-dz}{b+cz+dz^2},$$

and, by reading off coefficients,

$$\begin{aligned} [z^n] \frac{1}{1-z} F_m(z) &= \sum_{k=0}^n P_R(km) = \frac{a}{b+c+d} \\ &\quad + \frac{1}{b+c+d} (bP_R(m(n+1)) + dP_R(mn)). \end{aligned}$$

3. Conversion of powers

We need to rewrite $P_R(n)^m$ in a ‘linearized’ form. The cases m odd resp. even are significantly different in complexity.

$$P_R(n)^{2m+1} = \frac{2^{2m}}{R^{3m}(R+2)^m} \sum_{j=0}^m (-1)^j \binom{m}{j} P_R((m-2j)n) \left(-\frac{R}{2}\right)^{jn}.$$

The first one is basically a rearrangement following the binomial theorem:

$$\begin{aligned} (x - x^{-1})^{2m+1} &= \sum_{j=0}^{2m+1} \binom{2m+1}{j} (-1)^j x^{2j-2m-1} \\ &= \sum_{j=0}^m \binom{2m+1}{j} (-1)^j x^{2j-2m-1} + \sum_{j=m+1}^{2m+1} \binom{2m+1}{j} (-1)^j x^{2j-2m-1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^m \binom{2m+1}{j} (-1)^j x^{2j-2m-1} - \sum_{j=0}^m \binom{2m+1}{j} (-1)^j x^{-2j+2m+1} \\
&= \sum_{j=0}^m \binom{2m+1}{j} (-1)^j (x^{2j-2m-1} - x^{-2j+2m+1}).
\end{aligned}$$

The second one does not follow such a simple symmetry:

$$\begin{aligned}
P_R(n)^{2m} &= \frac{2^{2m+1}}{R^{3m}(R+2)^m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{P_R(2(m-j)(n+1))}{P_R(2(m-j))} \left(-\frac{R}{2}\right)^{jn} \\
&\quad - \frac{2^{2m-1}}{R^{3m-1}(R+2)^m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{P_R(2(m-j)n)Q_R(2(m-j))}{P_R(2(m-j))} \left(-\frac{R}{2}\right)^{jn} \\
&\quad + (-1)^m \binom{2m}{m} \frac{2^{2m}}{R^{3m}(R+2)^m} \left(-\frac{R}{2}\right)^{mn}.
\end{aligned}$$

This identity was found by creative guessing with a computer, taking two days. Using the Binet forms, this is basically a polynomial identity, like the one that we used earlier in [7]:

$$x^{2N} - x^{-2N} = (x + x^{-1}) \sum_{l=0}^{N-1} \binom{N+l}{N-l-1} (x - x^{-1})^{2l+1}.$$

We invite the readers to provide a formal proof, which should be basically an exercise in manipulating binomial coefficients. The creative part of this was to *find* the identity.

4. Summing powers

Using the formulae of the previous section, we can evaluate

$$\sum_{0 \leq k \leq n} P_R(\ell k)^m.$$

We have shown already how to sum

$$\sum_{k=0}^n P_R(km).$$

We need the slightly more general form, with a parameter σ ,

$$\sum_{k=0}^n P_R(km)\sigma^k,$$

because of the extra powers $(-R/2)^{jn}$ in our formulae. But that is not difficult, since in

$$\begin{aligned} F_m(z) &= \frac{2^m z P_R(m)}{2^m - z(R(R+1)2^{m-1}P_R(m) - R^2 2^{m-2}P_R(m-2)) + (-1)^m z^2 R^m} \\ &= \frac{az}{b + cz + dz^2} \end{aligned}$$

we just have to replace $z \rightarrow \sigma z$, and

$$\frac{1}{1-z} F_m(\sigma z) = \frac{a\sigma}{b + c\sigma + d\sigma^2} \frac{1}{1-z} - \frac{a\sigma}{b + c\sigma + d\sigma^2} \frac{b - d\sigma^2 z}{b + c\sigma z + d\sigma^2 z^2},$$

and, reading off coefficients again,

$$\begin{aligned} [z^n] \frac{1}{1-z} F_m(\sigma z) &= \frac{a\sigma}{b + c\sigma + d\sigma^2} \\ &\quad - \frac{1}{b + c\sigma + d\sigma^2} (-d\sigma^2 P_R(\sigma n) + b P_R(\sigma(n+1))). \end{aligned}$$

5. Examples

Here is the generating function of the numbers

$$\begin{aligned} &\sum_{k=0}^n P_R(k)^2 : \\ 16 &\frac{R-2}{(1-z)R^2(R+2)^2(3R-2)} - 16 \frac{1}{(zR+2)R^2(R+2)^2} \\ &\quad + 16 \frac{-zR + zR^2 + 2}{(R+2)(3R-2)R^2(z^2R^2 - 4zR(R+1) + 4)}. \end{aligned}$$

The coefficients can be expressed with generalized Pell numbers, as described in the previous sections.

Another example is this:

$$\sum_{k=0}^n P_R(2k)^3,$$

with generating function

$$\begin{aligned} & \frac{512(R^6 + 32R^4 + 32R^3 + 64)}{(R+2)^2(3R-2)(7R^2-2R+4)(3R^2+6R+4)} \\ & \quad \times \frac{1}{(R^2+2R-4)(R^3-4R^2-8)(z-1)} \\ & - \frac{384(zR^6-64)}{(z^2R^6-16zR^3(R+1)+64)R(R^3-4R^2-8)(R^2+2R-4)(R+2)^2} \\ & - \frac{128(2R+3)(2R+1)(zR^6-64)}{(z^2R^6-16zR^3(R+1)(4R^2+8R+1)+64)R(3R^2+6R+4)} \\ & \quad \times \frac{1}{(7R^2-2R+4)(3R-2)(R+2)^2}. \end{aligned}$$

We hope that this paper will trigger more research related to this new exciting family of numbers.

References

- [1] M. Bicknell, *A primer on the Pell sequence and related sequences*, Fibonacci Quart. **13** (1975), no. 4, 345–349.
- [2] D. Bród, *On a new one parameter generalization of Pell numbers*, Ann. Math. Sil. **33** (2019), 66–76.
- [3] J.M. Mahon and A.F. Horadam, *Ordinary generating functions for Pell polynomials*, Fibonacci Quart. **25** (1987), no. 1, 45–56.
- [4] *The online encyclopedia of integer sequences*. <http://oeis.org>.
- [5] T. Komatsu, *Higher-order identities for balancing numbers*, Notes Number Theory Discrete Math. **26** (2020), no. 2, 71–84.
- [6] T. Mansour and M. Shattuck, *Restricted partitions and q-Pell numbers*, Cent. Eur. J. Math. **9** (2011), no. 2, 346–355.
- [7] H. Prodinger, *On a sum of Melham and its variants*, Fibonacci Quart. **46/47** (2008/2009), no. 3, 207–215.
- [8] H. Prodinger, *Sums of powers over equally spaced Fibonacci numbers*, Integers **20** (2020), paper A37, 6 pp.

- [9] H. Prodinger, *How to sum powers of balancing numbers efficiently*, arXiv preprint 2020. Available at arXiv:2008.03916.
- [10] L. Trojnar-Spelina and I. Włoch, *On generalized Pell and Pell–Lucas numbers*, Iran. J. Sci. Technol. Trans. A Sci. **43** (2019), no. 6, 2871–2877.

DEPARTMENT OF MATHEMATICAL SCIENCES
STELLENBOSCH UNIVERSITY
7602 STELLENBOSCH
SOUTH AFRICA
e-mail: hproding@sun.ac.za