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ON APPROXIMATE CONFORMAL MAPPING OF A DISK AND AN ANNULUS WITH RADIAL AND CIRCULAR SLITS ONTO MULTIPLY CONNECTED DOMAINS

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Abstract. The method of boundary curve reparametrization is generalized to the case of multiply connected domains. We construct the approximate analytical conformal mapping of the unit disk with m circular slits and $n-m$ radial slits and an annulus with $(m-1)$ circular slits and $n-m$ radial slits onto an arbitrary given $(n+1)$ multiply connected finite domain with a smooth boundary. The method is based on extension of the Lichtenstein-Gershgorin equation to a multiply connected domain. The proposed method is reduced to the solution of a linear system with unknown Fourier coefficients. The approximate mapping function has the form of a Cauchy integral. Numerical examples demonstrate that the proposed method is effective in computations.

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1. Introduction

Conformal mappings by the analytical functions of a complex variable play an important role in the solution of many problems of mechanics and mathematics, particulary in the case of plane potential fields and the Laplace equation solution [1]. The conformal mapping of the circular domain (a disk with circular and radial slits or an annulus with circular and radial slits) onto a multiply connected domain can be applied to the solution of plane boundary value problems for corresponding domains by the Poincare series [2, 3] or equivalently by the Schottky--Klein prime function [4]. The existence and uniqueness of the solution for similar problems under certain assumptions is a consequence of the results of [5]. Computer progress has stimulated the appearance of many numerical methods for conformal mapping constructions [6,7]. For example, the widely-used Wegmann numerical method is based on the Riemann-Hilbert problem solution and involves iteration processes [8, 9]. Some authors searched for boundary reparametrization with linearization of this process as in [10, 11]. Many authors apply the integral equations which contain the singular integrals. The collocation method or Nyström's method can be applied for the solution of such integral equations [12].

Several types of canonical regions exist for conformal mappings [12]. Nasser managed to map bounded and unbounded multiply connected regions onto these five canonical regions by reformulating the mapping function as a Riemann-Hilbert problem which is solved by means of a boundary integral equation with the generalized Neumann kernel [12]. The right-hand side of the integral equation involves the integral with cotangent singularity which is approximated by Wittich's method. The integral equation was discretized by the Nyström method with the trapezoidal rule to obtain a linear system [12–18].

Here we continue the constructions of [19] and present a new method of the approximate conformal mapping of the unit disk with circular and radial slits and an annulus with circular and radial slits onto a multiply connected domain with a smooth boundary. We apply a method of integral equations generalising the Lichtenstein-Gershgorin one with a Neumann kernel obtained from the necessary and sufficient condition for a function defined at the points of a smooth contour to be the boundary values of some function analytical in the correspondent domain. These equations were thoroughly described in [7, 20]. We give the approximate solution of these equations by reduction to a linear system as in [13, 14, 19], therefore the method is easily programmable.

2. Approximate conformal mapping of the circular domains of two types onto a multiply connected domain by means of boundary reparametrization

Consider a finite $(n+1)$ - connected domain D_z bounded by the outer simple smooth curve L_0 and the inner simple smooth curves L_s given by the equations

$$L_s = \{z = z_s(t),\ z_s(0) = z_s(2\pi),\ t \in [0, 2\pi]\},\ s = 0, \dots, N.$$

We also assume that the boundary curves L_s complex representations are as follows:

$$z_s(t) = \sum_{k=-m_s}^{n_s} d_{ks} e^{ikt},\ t \in [0, 2\pi],\ s = 0, \dots, N.$$

The parametrization traces the domain D_z along L_0 counterclockwise and along the inner contours L_s, $s = 1, \dots, n$, clockwise.

Definition. We call the unit disk with m circular slits $\zeta = R_s e^{i\theta}$, $\theta_{1s} < \theta < \theta_{2s}$, $0 < R_s < 1$, $\theta_{2s} - \theta_{1s} < 2\pi$, $s = 1, \dots, m$, and with $n - m$ radial slits $\zeta = Re^{i\theta_j}$, $0 < R_{1j} < R < R_{2j} < 1$, $0 < \theta_j < 2\pi$, $j = m+1, \dots, n$, an $(n+1)$ - connected canonical domain of the first type. We call the annulus with the exterior radius 1, with the interior radius r, $r < 1$, and with $(m-1)$ circular slits $\zeta = R_s e^{i\theta}$, $\theta_{1s} < \theta < \theta_{2s}$,

$r < R_s < 1$, $\theta_{2s} - \theta_{1s} < 2\pi$, $s = 1,\dots,m-1$, and with $n-m$ radial slits $\zeta = Re^{i\theta_j}$, $0 < R_{1j} < R < R_{2j} < 1$, $0 < \theta_j < 2\pi$, $j = m+1,\dots,n$, an $(n+1)-$ connected canonical domain of the second type.

Theorem 1 *Analytic-numerical approximate conformal mapping exists for an $(n+1)$-connected circular domain D_ζ of the first type such that the function $f(\zeta)$ maps conformally the domain D_ζ onto the given $(n+1)$-connected domain D_z with smooth boundary components. The approximate solution converges to the exact one as $O(1/N^2)$. Here N is the size of the truncated auxiliary matrix.* □

PROOF Existence of solution is a known fact [5], so we concentrate on the approximate solution construction and proof of its convergence. The map is unique under the following conditions: $f(0) = A+iB$, $f(1) = C+iD$, $(A,B) \in D_z$, $(C,D) \in L_0$.

We assume that $0 \in D_z$ and $A+iB = 0$ without loss of generality. We give the constructive proof. We construct the conformal map of the circular domain of the first type onto the domain D_z by reparametrization of the given boundary representations. So we search for the function $t_0(\theta)$, $\theta \in [0,2\pi]$, for the functions $t_s(\theta)$, $s = 1,\dots,m$, $\theta \in [\theta_{1s},\theta_{2s}]$, and for the functions $t_j(R)$, $R \in [R_{1j},R_{2j}]$, such that the values $z_s(t_s(\theta))$, $s = 0,\dots,m$, $z_j(t_j(R))$, be the boundary values of an analytic function in the corresponding circular domain. The parameters R_s, θ_{1s}, θ_{2s}, $s = 1,\dots,m$, R_{1j}, R_{2j}, θ_j, $j = m+1,\dots,n$, are also unknown and will be found within the solution process.

Let us consider the analytic in the domain D_z function $\zeta(z)$ which maps conformally the domain D_z onto D_ζ with the correspondence $\zeta(0) = 0$ and the analytic in D_z function $\log\dfrac{z}{\zeta}$. According to [21], the necessary and sufficient condition for $\log\dfrac{z}{\zeta}$ to be analytic in D_z are the boundary relations

$$\log\frac{z_s(t)}{R_s e^{i\theta_s(t)}} = \sum_{\sigma=0}^{m}\frac{1}{\pi i}\int_0^{2\pi}\log\frac{z_\sigma(\tau)}{R_\sigma e^{i\theta_\sigma(\tau)}}[\log(z_\sigma(\tau)-z_s(t))]'_\tau d\tau +,$$

$$\sum_{j=m+1}^{n}\frac{1}{\pi i}\int_0^{2\pi}\log\frac{z_j(\tau)}{R_j(\tau)e^{i\theta_j}}[\log(z_j(\tau)-z_s(t))]'_\tau d\tau, \tag{1}$$

where $t \in [0,2\pi]$, $s = 0,\dots,m$, $R_0 = 1$, and

$$\log\frac{z_j(t)}{R_j(t)e^{i\theta_j}} = \sum_{s=0}^{m}\frac{1}{\pi i}\int_0^{2\pi}\log\frac{z_s(\tau)}{R_s e^{i\theta_s(\tau)}}[\log(z_s(\tau)-z_j(t))]'_\tau d\tau +,$$

$$\sum_{k=m+1}^{n}\frac{1}{\pi i}\int_0^{2\pi}\log\frac{z_k(\tau)}{R_k(\tau)e^{i\theta_k}}[\log(z_k(\tau)-z_j(t))]'_\tau d\tau, \tag{2}$$

where $t \in [0, 2\pi]$, $j = m+1, \ldots, n$.

We introduce the following functions: $q_s(t) = \arg z_s(t) - \theta_s(t)$, where $\theta_s(t)$ is the polar angle of the image of the point of $z_s(t)$, $s = 0, \ldots, m$, and $p_j(t) = \log|z_j(t)| - \log R_j(t)$, where $R_j(t)$ is the radius of the image of the point of $z_j(t)$, $j = m+1, \ldots, n$.

We separate the imaginary part of both sides of equation (1) and arrive to the equation system described in [3]:

$$q_s(t) = \sum_{\sigma=0}^{m} \frac{1}{\pi} \int_0^{2\pi} q_\sigma(\tau)[\arg(z_\sigma(\tau) - z_s(t))]'_\tau d\tau - \sum_{\sigma=0}^{m} \frac{1}{\pi} \int_0^{2\pi} \log \frac{|z_\sigma(\tau)|}{R_\sigma} [\log|z_\sigma(\tau) - z_s(t)|]'_\tau d\tau +$$

$$+ \sum_{j=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} [\arg z_j(\tau) - \theta_j][\arg(z_j(\tau) - z_s(t))]'_\tau d\tau -$$

$$- \sum_{j=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} p_j(\tau)[\log|z_j(\tau) - z_s(t)|]'_\tau d\tau, \quad s = 0, \ldots, m. \tag{3}$$

This system generalizes the Lichtenstein-Gershgorin equation for non simple-connected regions.

We separate the real part of both sides of equation (2):

$$p_j(t) = \sum_{s=0}^{m} \frac{1}{\pi} \int_0^{2\pi} [\log|z_s(\tau)| - \log R_s][\arg(z_s(\tau) - z_j(t))]'_\tau d\tau +$$

$$+ \sum_{s=0}^{m} \frac{1}{\pi} \int_0^{2\pi} q_s(\tau)[\log|z_s(\tau) - z_j(t)|]'_\tau d\tau + \sum_{k=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} p_k(\tau)[\arg(z_k(\tau) - z_j(t))]'_\tau d\tau +$$

$$+ \sum_{k=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} [\arg[z_k(\tau) - \theta_k][\log|z_k(\tau) - z_j(t)|]'_\tau d\tau, \quad j = m+1, \ldots, n. \tag{4}$$

After differentiating relations (3) and (4) with respect to t and integrating the results by parts, we obtain the following relations on the functions $q'_s(t)$ and $p'_j(t)$ respectively:

$$q'_s(t) = \sum_{\sigma=0}^{m} \frac{1}{\pi} \int_0^{2\pi} q'_\sigma(\tau) K_{\sigma,s}(\tau, t) d\tau + \sum_{j=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} p'_j(\tau) L_{j,s}(\tau, t) d\tau + Q_s(t), \; s = 0, \ldots, m, \tag{5}$$

$$p'_j(t) = -\sum_{s=0}^{m} \frac{1}{\pi} \int_0^{2\pi} q'_s(\tau) L_{s,j}(\tau,t)d\tau + \sum_{k=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} p'_k(\tau) K_{k,j}(\tau,t)d\tau + P_j(t), \;\; j = m+1,\dots,n, \tag{6}$$

where

$$K_{\sigma s}(\tau,t) = -[\arg(z_\sigma(\tau) - z_s(t))]'_t, \;\; L_{j,s}(\tau,t) = [\log(z_j(\tau) - z_s(t))]'_t,$$

$$Q_s(t) = \sum_{\sigma=0}^{m} \frac{1}{\pi} \int_0^{2\pi} [\log|z_\sigma(\tau)|]' L_{\sigma,s}(\tau,t)d\tau + \sum_{j=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} (\arg z_j(\tau))' K_{j,s}(\tau,t)d\tau,$$

$$P_j(t) = \sum_{s=0}^{m} \frac{1}{\pi} \int_0^{2\pi} [\log|z_s(\tau)|]' K_{s,j}(\tau,t)d\tau - \sum_{k=m+1}^{n} \frac{1}{\pi} \int_0^{2\pi} (\arg z_k(\tau))' L_{k,j}(\tau,t)d\tau.$$

The kernel $L_{\sigma,s}$ has a singularity in the form of $\cot\frac{\tau - t}{2}$ for $\sigma = s$:

$$(\log|z_s(\tau) - z_s(t)|)'_t = \Re\left(\log \sum_{k=-m_s}^{n_s} d_{ks}[e^{ik\tau} - e^{ikt}]\right)'_t = \Re\left(\log\sin\frac{\tau - t}{2} + \right.$$

$$\left. + \log\left[\sum_{k=1}^{n_s} d_{ks} e^{ikt} \sum_{l=0}^{k-1} e^{il(\tau - t)} - \sum_{k=1}^{m_s} d_{(-k)s} e^{-ik\tau} \sum_{l=0}^{k-1} e^{il(\tau - t)}\right]\right)'_t =$$

$$= -\frac{1}{2}\cot\frac{\tau - t}{2} + \left(\log\left|\sum_{k=1}^{n_s} d_{ks} e^{ikt} \sum_{l=0}^{k-1} e^{il(\tau - t)} - \sum_{k=1}^{m_s} d_{(-k)s} e^{-ik\tau} \sum_{l=0}^{k-1} e^{il(\tau - t)}\right|\right)'_t.$$

The Cauchy principal value integral $\frac{1}{\pi}\int_0^{2\pi} [\log|z_\sigma(\tau)|]' \cot\frac{\tau - t}{2} d\tau$ can be calculated via Hilbert formula [15] as in [14]. Finally, we obtain the following system of Fredholm integral equations of the second kind which can be written in the operator form as follows:

$$\begin{pmatrix} I - K_{0,0} & -K_{1,0} & \dots & -K_{m,0} & -L_{m+1,0} & \dots & -L_{n,0} \\ -K_{0,1} & I - K_{1,1} & \dots & -K_{m,1} & -L_{m+1,1} & \dots & -L_{n,1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -K_{0,m} & -K_{1,m} & \dots & I - K_{m,m} & -L_{m+1,m} & \dots & -L_{n,m} \\ L_{0,m+1} & L_{1,m+1} & \dots & L_{m,m+1} & I - K_{m+1,m+1} & \dots & -K_{n,m+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ L_{0,n} & L_{1,n} & \dots & L_{m,n} & -K_{m+1,n} & \dots & I - K_{n,n} \end{pmatrix} \begin{pmatrix} q'_0 \\ \dots \\ q'_m \\ p'_{m+1} \\ \dots \\ p'_n \end{pmatrix} =$$

$$= \left(Q_0, \dots, Q_m, P_{m+1}, \dots, P_n\right)^T$$

The last operator system can be reduced to the infinite linear system over the Fourier coefficients of the unknown functions $q_s'(t)$, $s = 0, \dots, m$, $p_j'(t)$, $j = m + 1, \dots, n$, if we find the coefficients of double Fourier expansions of the kernels of integral operators and compare the coefficients with the same trigonometric functions [13]. The approximate solution of the infinite system over Fourier coefficients of the unknown functions is a solution of a truncated system over the Fourier coefficients of the unknown functions.

Convergence of the approximate solution of system (5)-(6) to the exact one provided $M \to \infty$ was proved in [13] for the case of conformal mapping of a simply connected domain. This proof can be applied to the case of multiply connected domain if we replace the corresponding space l^2 by the space $l^2 \times l^2 \times ... \times l^2$.

We search for the approximate solution of system (5)-(6) in the form of Fourier polynomials:

$$q_s'(t) = \sum_{l=1}^{M} \alpha_{ls} \cos lt + \beta_{ls} \sin lt, \;\; s = 0, \dots, m, p_j'(t) = \sum_{l=1}^{M} \alpha_{lj} \cos lt + \beta_{lj} \sin lt, \quad (7)$$

Here $j = m+1, \dots, n$, $t \in [0, 2\pi]$.

Now integral Fredholm equations of the second kind in (5) and (6) can be reduced to the linear system over Fourier coefficients α_{ls} and β_{ls}, $s = 0, \dots, m$, α_{lj} and β_{lj}, $j = m+1, \dots, n$:

$$\begin{pmatrix} A_{00} & B_{00} & A_{01} & \dots & B_{0n} & E_{0m+1} & F_{0m+1} & \dots & E_{0n} \\ C_{00} & D_{00} & C_{01} & \dots & D_{0m} & G_{0m+1} & H_{0m+1} & \dots & H_{0n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ C_{m0} & D_{m0} & C_{m1} & \dots & D_{mm} & G_{mm+1} & H_{mm+1} & \dots & H_{mn} \\ P_{m+10} & Q_{m+10} & P_{m+11} & \dots & Q_{m+1m} & R_{m+1m+1} & S_{m+1m+1} & \dots & S_{m+1n} \\ N_{m+10} & T_{m+10} & N_{m+11} & \dots & T_{m+1m} & V_{m+1m+1} & U_{m+1m+1} & \dots & U_{m+1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ N_{n0} & T_{n0} & N_{n1} & \dots & T_{nm} & V_{nm+1} & U_{nm+1} & \dots & U_{nn} \end{pmatrix} \times \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \alpha_1 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} =$$

$$= \left(a_0, b_0, a_1, b_1, \dots, b_n\right)^T,$$

where $\alpha_s = (\alpha_{1s}, \dots, \alpha_{ns})^T$, $\beta_s = (\beta_{1s}, \dots, \beta_{ns})^T$. The vectors $a_s = (a_{1s}, \dots, a_{ns})^T$, $b_s = (b_{1s}, \dots, b_{ns})^T$ on the right-hand side of the system consist of the elements

$$a_{js} = \frac{1}{\pi} \int_0^{2\pi} Q_s(t) \cos jt dt, \; b_{js} = \frac{1}{\pi} \int_0^{2\pi} Q_s(t) \sin jt dt, \;\; j = 1, \dots, m, \; s = 0, \dots, m,$$

$$a_{jk}=\frac{1}{\pi}\int_0^{2\pi}P_k(t)\cos jtdt,\ b_{jk}=\frac{1}{\pi}\int_0^{2\pi}P_k(t)\sin jtdt,\quad j=1,\dots,m,\ k=m+1,\dots,n,$$

The block matrices $A_{\sigma s}$, $B_{\sigma s}$, $C_{\sigma s}$, $D_{\sigma s}$, $E_{\sigma l}$, $F_{\sigma l}$, $G_{\sigma l}$, $H_{\sigma l}$, P_{ps}, Q_{ps}, N_{ps}, T_{ps}, R_{pl}, S_{pl}, V_{pl}, U_{pl}, $\sigma,s=0,\dots,m$, $l,p=m+1,\dots,n$ of size $M\times M$ consist of the elements

$$A_{\sigma sjk}=\delta_{\sigma s}\delta_{jk}-\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}K_{\sigma s}(\tau,t)\cos jtdt, B_{\sigma sjk}=-\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}K_{\sigma s}(\tau,t)\cos jtdt,$$

$$C_{\sigma sjk}=-\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}K_{\sigma s}(\tau,t)\sin jtdt, D_{\sigma sjk}=\delta_{\sigma s}\delta_{jk}-\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}K_{\sigma s}(\tau,t)\sin jtdt,$$

$$E_{\sigma ljk}=-\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}L_{\sigma l}(\tau,t)\cos jtdt, F_{\sigma ljk}=-\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}L_{\sigma l}(\tau,t)\cos jtdt,$$

$$G_{\sigma ljk}=-\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}L_{\sigma l}(\tau,t)\sin jtdt, U_{pljk}=\delta_{pl}\delta_{jk}-\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}K_{pl}(\tau,t)\sin jtdt,$$

$$H_{\sigma ljk}=-\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}L_{\sigma l}(\tau,t)\sin jtdt, P_{psjk}=\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}L_{ps}(\tau,t)\cos jtdt,$$

$$Q_{psjk}=\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}L_{ps}(\tau,t)\cos jtdt, N_{psjk}=\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}L_{ps}(\tau,t)\sin jtdt,$$

$$T_{psjk}=\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}L_{ps}(\tau,t)\sin jtdt, R_{pljk}=\delta_{pl}\delta_{jk}-\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}K_{pl}(\tau,t)\cos jtdt,$$

$$S_{pljk}=-\frac{1}{\pi^2}\int_0^{2\pi}\sin k\tau d\tau\int_0^{2\pi}K_{pl}(\tau,t)\cos jtdt, V_{pljk}=-\frac{1}{\pi^2}\int_0^{2\pi}\cos k\tau d\tau\int_0^{2\pi}K_{pl}(\tau,t)\sin jtdt,$$

where $j,k=1,\dots,m$, δ_{rt} is the Kronecker delta function.

The functions $q_s(t)$, $s=0,\dots,m$, and $p_j(t)$, $j=m+1,\dots,n$, can be restored via their derivatives (7) with an arbitrary constant summand

$$q_s(t)=q_{0s}+\tilde{q}_s(t),\ \ \tilde{q}_s(t)=\sum_{l=1}^{M}\frac{\alpha_{ls}}{l}\sin lt-\frac{\beta_{ls}}{l}\cos lt,$$

$$p_j(t)=p_{0j}+\tilde{p}_j(t),\ \ \tilde{p}_j(t)=\sum_{l=1}^{M}\frac{\alpha_{lj}}{l}\sin lt-\frac{\beta_{ls}}{l}\cos lt,\ \ t\in[0,2\pi]. \tag{8}$$

We choose the constant summand q_{00} in accordance with the condition $f(1)=C+iD$ in the following way. We find the value of the parameter $\hat{t}$ such that $z_0(\hat{t})=C+iD$. Now $q_{00}=\arg(C+iD)-\tilde{q}_0(\hat{t})$.

We obtain the values of the other constant summands q_{0s}, $s=1,\dots,m$, p_{0j}, $j==m+1,\dots,n$, and also the values of R_s, $s=1,\dots,m$, θ_j, $j=m+1,\dots,n$, in the following way. We take n points in each of n finite component of the set complement of D_z. Let us denote these points z_k^*, $k=1,\dots,n$. The function $\log\dfrac{z_s(t)}{R_s e^{i\theta(t)}}$, $s=0,\dots,m$, $\log\dfrac{z_j(t)}{R(t)e^{i\theta_j(t)}}$, $j=m+1,\dots,n$, is the boundary value of the analytical in D_z function, so the Cauchy integral with the corresponding density along the boundary of D_z vanishes at the points z_j^*, $j=1,\dots,n$. Therefore, we have the linear complex system

$$\sum_{\sigma=0}^{m}\frac{1}{2\pi i}\int_0^{2\pi}[iq_{0\sigma}-\log R_\sigma+\log|z_\sigma(\tau)|+i\tilde{q}_s(\tau)]\,[\log(z_\sigma(\tau)-z_j^*)]'_\tau d\tau+$$

$$\sum_{k=m+1}^{n}\frac{1}{2\pi i}\int_0^{2\pi}[-i\theta_k+i\arg z_k(\tau)+p_{0k}+\tilde{p}_k(\tau)]\,[\log(z_k(\tau)-z_j^*)]'_\tau d\tau=0,\ \ j=1,\dots,n,$$

with the unknown real $q_{0\sigma}$, $\log R_\sigma$, $\sigma=1,\dots,m$, p_{0k}, θ_k, $k=m+1,\dots,n$.

We restore the values of θ_{1j} , θ_{2j}, $j=1,\dots,m$, after we have restored q_{0j}. Then

$$\theta_{1j}=\min_{t\in[0,2\pi]}[\arg z_i(t)-\tilde{q}_j(t)]-q_{0j},\ \theta_{2j}=\max_{t\in[0,2\pi]}[\arg z_i(t)-\tilde{q}_j(t)]-q_{0j}.$$

We restore the values of R_{1j} , R_{2j}, $j=m+1,\dots,n$, after we have restored p_{0j}. Then

$$\log R_{1j}=\min_{t\in[0,2\pi]}[\log|z_j(t)|-\tilde{p}_j(t)]-p_{0j},\ \log R_{2j}=\max_{t\in[0,2\pi]}[\log|z_j(t)|-\tilde{p}_j(t)]-p_{0j}.$$

So all parameters of the circular domain of the first type D_ζ have been found.

Now we have the functions $q_s(t)$, $s=0,\dots,m$, $p_k(t)$, $k=m+1,\dots,n$, $t\in[0,2\pi]$, and therefore we can restore the relations between the boundary parameters of the domains D_z and D_ζ via the formulas $\theta_s(t)=\arg z_s(t)-q_s(t)$, $s=0,\dots,m$, $\log R_k(t)=$

$= \log|z_k(t)| - p_k(t)$, $k = m+1,\dots,n$. Note that $\theta_0(t)$ grows monotonically when t grows from 0 to 2π, $\theta_0(2\pi) - \theta_0(0) = 2\pi$, while each of the functions $\theta_s(t)$, $s = = 1,\dots,m$, $\log R_k(t)$, $k = m+1,\dots,n$, is 2π-periodic with one interval of increase and one interval of decrease. We can restore the inverse to $\theta_0(t)$ monotonically increasing function $t_0(\theta)$ and we can restore the single-valued functions $t_s^{\pm}(\theta)$, $\theta \in [\theta_{1s}, \theta_{2s}]$, $s = 1,\dots,m$, and $t_s^{\pm}(R)$, $R \in [R_{1k}, R_{2k}]$, $k = m+1,\dots,n$.

The approximate analytical function which maps D_ζ onto D_z now has the form of the Cauchy integral

$$f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z_0(t_0(\theta))e^{i\theta}d\theta}{e^{i\theta}-\zeta} + \sum_{s=1}^{m} \frac{1}{2\pi}\int_{\theta_{1s}}^{\theta_{2s}} \frac{[z_s(t_s^+(\theta)) - z_s(t_s^-(\theta))]R_s e^{i\theta}}{R_s e^{i\theta}-\zeta}d\theta +$$

$$+ \sum_{j=m+1}^{n} \frac{1}{2\pi i}\int_{R_{1j}}^{R_{2s}} \frac{[z_j(t_j^+(R)) - z_j(t_j^-(R))]e^{i\theta_j}}{Re^{i\theta_j}-\zeta}dR.$$

We can apply the Cauchy integral in the form

$$f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z_0(t)e^{i\theta_0(t)}\theta_0'(t)dt}{e^{i\theta_0(t)}-\zeta} + \sum_{s=1}^{m} \frac{1}{2\pi}\int_0^{2\pi} \frac{z_s(t)R_s e^{i\theta_s(t)}\theta_s'(t)}{R_s e^{i\theta_s(t)}-\zeta}dt +$$

$$+ \sum_{k=m+1}^{n} \frac{1}{2\pi i}\int_0^{2\pi} \frac{z_j(t)R_j'(t)e^{i\theta_j}}{R_j(t)e^{i\theta_j}-\zeta}dt. \tag{9}$$

in order not to deal with the functions $t_s^{\pm}(\theta)$ or $t_j^{\pm}(R)$ and not to integrate along the different borders of the same slit.

The values of $f(\zeta)$ at the points of D_ζ close to the boundary can be calculated with the help of analytic continuation of the Cauchy integral as in [22].

Theorem 2 *Analytic-numerical approximate conformal mapping exists for an $(n+1)$-connected circular domain D_ζ of the second type such that the function $f(\zeta)$ maps conformally the domain D_ζ onto the given $(n+1)$-connected domain D_z with smooth boundary components. The approximate solution converges to the exact one as $O(1/N^2)$. Here N is the size of the truncated auxiliary matrix.* □

PROOF Existence of the exact solution is a generalisation of the Riemann Theorem [5]. The map is unique under the following conditions: the image of the inner circle $|\zeta| = r$ is the boundary component L_j, $j \in \{1, 2, \dots, m\}$, $f(1) = C + iD$, $(C, D) \in L_0$.

We assume that $j = 1$ and $\int_0^{2\pi} (\arg z_1(t))' dt = -2\pi$ without loss of generality. We construct the conformal map of the circular domain of the first type onto the domain

D_z by reparametrization of the given boundary representations. So, we search for functions $t_s(\theta)$, $\theta \in [0,2\pi]$, $s=0,1$, for the functions, for the functions $t_s(\theta)$, $\theta \in [\theta_{1s},\theta_{2s}]$, $s=2,\dots,m$, and for the functions $t_j(R)$, $R \in [R_{1j},R_{2j}]$, $j=m+1,\dots,n$. The construction is as the one for mapping of the circular domain of the first type.

We consider the analytic in the domain D_z function $\zeta(z)$ which maps conformly the domain D_z onto D_ζ and the analytic in D_z function $\log\frac{z}{\zeta}$. We apply the necessary and sufficient conditions for $\log\frac{z}{\zeta}$ to be analytic in D_z which are boundary relations (1) and (2) as above. We introduce the functions $q_s(t)=\arg z_s(t)-\theta_s(t)$, $s=0,\dots,m$ and $p_j(t)=\log|z_j(t)|-\log R_j(t)$, $j=m+1,\dots,n$. After separation of the imaginary or real parts of both sides of these equations, differentiation with respect to τ and integration by parts we have equations (3) and (4). We reduce the solution of the integral equations to the solution of a linear system with truncated matrices if we consider $q_s'(t)$ representations (7). Now we restore the functions $q_s(t)$, $p_j(t)$ as in (8). The constant summand q_{00} can be restored in the same way as for the previous case. The values of q_{0s}, p_{0j}, and R_s, θ_j, $s=1,\dots,m$, $j=m+1,\dots,n$, can also be restored as above with the help of the additional points z_s^*, $s=1,\dots,n$, located in the exterior of the domain D_z. Note that $z_1^*=0$. ■

3. Examples

1. Elliptic domain with two elliptic holes. Consider the elliptic domain bounded by the curve $6e^{it}-e^{-it}$ with the holes bounded by the curves $0.2e^{it}-0.8e^{-it}-3i$ and $0.8e^{i(-t+\pi/4)}+0.2e^{-i(-t-\pi/4)}+3i$, $t\in[0,2\pi]$. We constructed the conformal mapping from the unit disk with two radial slits: over the interval $[0.25e^{-i1.578},0.82e^{-i1.578}]$ and over the interval $[0.27e^{i1.588},0.8e^{i1.588}]$, the former is the preimage of the lower ellipse and the latter is mapped into the upper one. We give the preimage of the domain, a part of the polar coordinate net and the solution result (Fig. 2).

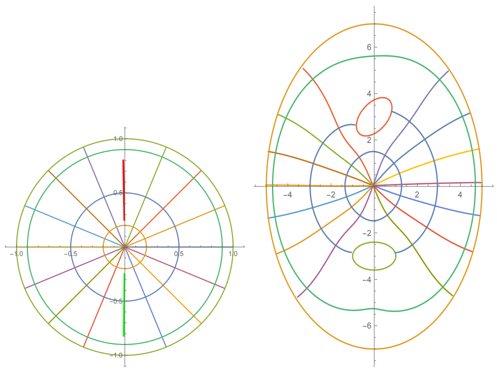

Fig. 1. The elliptic domain with two holes and radial slits

2. Elliptic domain with two elliptic holes. Consider the elliptic domain bounded by the curve $6e^{it} - e^{-it}$ with the holes bounded by the curves $0.2e^{it} - 0.8e^{-it} - 3i$ and $0.8e^{i(-t+\pi/4)} + 0.2e^{-i(-t-\pi/4)} + 3i, t \in [0, 2\pi]$. We constructed the conformal mapping from the unit disk with one radial and one circular slit: over the interval $[e^{0.93i}, e^{2.18i}]$ and over the interval $[0.27e^{-i1.588}, 0.84e^{-i1.588}]$, the former is the preimage of the lower ellipse and the latter is mapped into the upper one. We give the preimage of the domain, a part of the polar coordinate net and the solution result (Fig. 2).

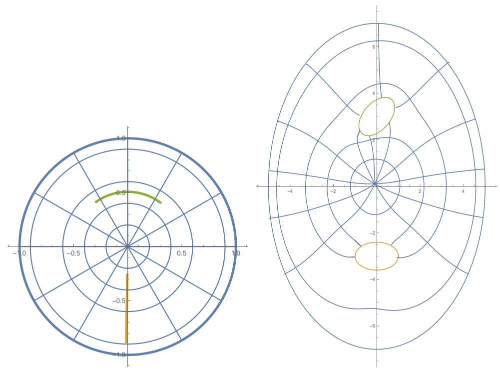

Fig. 2. The elliptic domain with two holes; circular and radial slits

4. Conclusions

Our method of the approximate solution construction is based on linear integral equations. These equations are reduced to an infinite linear system over Fourier coefficients of unknown conjugate functions. The infinite system is truncated to a finite one. The method does not apply any auxiliary constructions or specific conformal mappings, it does not use the accessory solutions of boundary value problems and it does not require iterations. It allows us to construct and compute the conformal mappings onto different multi-connected domains without recursion and other iterative procedures. The accuracy similar to that of [23] can be achieved with a matrix of the size of 200. The resulting approximate solution presented here is an analytic function with all of its properties such as possessing derivatives of any order. The natural replacement of all the features in our case is a solution of a large linear system.

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