

## COMPLEX GLEASON MEASURES AND THE NEMYTSKY OPERATOR

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**Abstract.** This work is devoted to the generalization of previous results on Gleason measures to complex Gleason measures. We develop a functional calculus for complex measures in relation to the Nemytsky operator. Furthermore we present and discuss the interpretation of our results with applications in the field of quantum mechanics. Some concrete examples and further extensions of several theorems are also presented.

### 1. Introduction

Gleason Measures have been deeply studied for their close relationship with the foundations of Quantum Mechanics. In fact, Gleason measures have a natural quantum mechanical interpretation (see [15]) and it is for this reason that many authors have dedicated their research to Gleason's Theorem and its generalization (for instance [1]-[13],[20],[31]) as well as to application problems in quantum mechanics such as the one related to hidden variables (see [21],[9]). In particular the vectorial character of the complex Gleason measures gives a natural interpretation to such mathematical concept; many quantities considered in physics have a vectorial character as well; the momentum and the angular momentum of a particle exemplify this concept, and in quantum mechanics, this kind of quantities are represented by what we call a vector

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operator. All these relations have led us to generalizing previous results on Gleason Measures to complex Gleason measures.

With the purpose of introducing this concept, consider for example the momentum of a particle. In order to perform a measurement of a particle's momentum in two dimensions, we need to measure the two components of the momentum  $p_x, p_y$  relative to an (arbitrarily chosen) orthogonal coordinate system. Accordingly, in quantum mechanics, the momentum is represented by two observables (self-adjoint operators)  $P_x$  and  $P_y$ , and we may think of these two operators as the “components” of one vector operator:

$$P = (P_x, P_y)$$

relative to the coordinate system chosen previously. This means that if a linear change of coordinates is made, the components of  $P$  change according to the same rule as the components of a vector; but the momentum itself should be independent of the coordinate system chosen (otherwise it would not have a physical meaning).

To stress this independence of the coordinate system, we shall follow a different point of view: Notice that in order to measure a vector quantity like  $p$ , it is necessary to be able to measure the scalar product  $\langle p, v \rangle$  with any given direction  $v$ . According to the quantum mechanics formalism we would have an observable (self-adjoint operator)  $P_v$  for each  $v$ . Furthermore the correspondence  $v \rightarrow P_v$  should be linear. As a consequence, we make the following formal definition: Let  $H$  be a Hilbert space (to be thought as the state space of the quantum system) and  $V$  a real Hilbert space (the space of values of the vectorial quantity that we want to measure); a vectorial operator is an element of  $\mathcal{L}(V, \mathcal{L}(H))$ .

The Nemytsky operator is a variable-coefficient composition operator of the form  $\varphi(x) \rightarrow g(x, \varphi(x))$  that has been studied and used in the context of many nonlinear problems involving integrals, as well as partial and ordinary differential equations (see for instance, [4], [5], [7], [8], [18], [22], [23], [26], [27] and [37]).

In [4], the Nemytsky operator was extended to signed measures, using this extension to solve an initial value problem with a signed finite measure as initial condition, for a class of nonlinear evolution equations. The aim was to develop a functional calculus for signed measures that allow one to give a meaning to nonlinear term under fairly general conditions. The present article develops a functional calculus that permits working with complex Gleason measures.

We organize our work as follows. In section 2 notations, definitions and results that will be used in the paper are introduced. In sections 3, 4, 5 and 6 we generalize previous results to complex Gleason measures. Sections 7 and 8 are devoted to the Nemytsky operator. In section 9 we present our

main result; *A Gleason measure can be used as an operator measure for the Nemytsky operator*. Finally in section 10 we present applications and examples in the field of quantum mechanics.

## 2. Preliminary definitions and results

We shall begin this section by defining some Dirac notations, which for convenience will be used in some sections of this paper.

A quantum state of a particle will be characterized by a state vector, belonging to an abstract space  $\mathcal{E}_r$  called the state space of a particle. The fact that the space  $\mathcal{F}$  is a subspace of  $L^2$  implies that  $\mathcal{E}_r$  is a subspace of a Hilbert space. We will define the notation and the rules of vector calculations in  $\mathcal{E}_r$ . Before we proceed, we state the following postulate: The quantum state of any physical system is characterized by a state vector, belonging to a space  $\mathcal{E}$  which is the state space of the system.

**DEFINITION 2.1.** Any element, or vector of the space  $\mathcal{E}$  is called a *ket vector* or a *ket*. It is symbolized by  $|\rangle$ , inside which is placed a unique sign which enables us to distinguish the corresponding ket from all others, for example  $|\psi\rangle$ .

**DEFINITION 2.2.** A linear functional  $\chi$  is a linear operation which associates a complex number with every ket  $|\psi\rangle$ .

We remark that the set of linear functionals defined on the kets  $|\psi\rangle \in \mathcal{E}$  constitutes a vector space, which is called the dual space of  $\mathcal{E}$  and is represented as  $\mathcal{E}^*$ .

Next, we describe the elements of the dual space  $\mathcal{E}^*$  of  $\mathcal{E}$ .

**DEFINITION 2.3.** Any element, or vector of  $\mathcal{E}^*$ -space is called a *bra vector* or a *bra*. It is represented by the symbol  $\langle |$ . For example, the bra  $\langle \chi|$  designates the linear functional  $\chi$  and we shall henceforth use the notation  $\langle \chi|\psi\rangle$  to denote the complex number obtained by causing the linear functional  $\langle \chi| \in \mathcal{E}^*$  to act on the ket  $|\psi\rangle \in \mathcal{E}$ :

$$\chi(|\psi\rangle) = \langle \chi|\psi\rangle.$$

The existence of a scalar product in  $\mathcal{E}$  will enable us to show that we can associate, with every ket  $|\varphi\rangle \in \mathcal{E}$ , an element of  $\mathcal{E}^*$ , that is, a bra, which is

denoted by  $\langle\varphi|$ . Suppose  $\langle\varphi|$  is a linear functional, the scalar product is defined by the relation

$$\langle\varphi|\psi\rangle = (|\varphi\rangle, |\psi\rangle).$$

Now assume that we write  $\langle\varphi|$  and  $|\psi\rangle$  in the opposite order:

$$(2.1) \quad |\psi\rangle\langle\varphi|.$$

We shall observe that if we abide by the rule of juxtaposition of symbols, the expression (2.1) represents an operator. For example select an arbitrary ket say  $|\chi\rangle$  and consider the expression

$$(2.2) \quad |\psi\rangle\langle\varphi|\chi\rangle.$$

We know that  $\langle\varphi|\chi\rangle$  is a complex number and so (2.2) is a ket, obtained by multiplying  $|\psi\rangle$  by the scalar  $\langle\varphi|\chi\rangle$ . Thus  $|\psi\rangle\langle\varphi|$  applied to an arbitrary ket, gives another ket hence it is an operator.

Next we define a projector  $S_\psi$  onto a ket  $|\psi\rangle$ .

Let  $|\psi\rangle$  be a ket which is normalized to one, i.e.  $\langle\psi|\psi\rangle = 1$ . Consider the operator  $S_\psi = |\psi\rangle\langle\psi|$  and apply it to an arbitrary ket  $|\varphi\rangle$ :

$$S_\psi|\varphi\rangle = |\psi\rangle\langle\psi|\varphi\rangle.$$

$S_\psi$  acting on an arbitrary ket  $|\varphi\rangle$ , gives a ket proportional to  $|\psi\rangle$ . In fact the coefficient of proportionality  $\langle\psi|\varphi\rangle$  is the scalar product of  $|\varphi\rangle$  by  $|\psi\rangle$ . Therefore  $S_\psi$  is the orthogonal projection operator onto the ket  $|\psi\rangle$ . Please refer to [13] for more details of the dirac notation.

In subsequent parts of this section we will briefly review some definitions and properties related to vector valued measures. The measurability of vector valued functions are also presented (for more details see [16]). We assume that  $(S, \Sigma, \mu)$  is a complete  $\sigma$ -finite measure space and  $X$  is a real Banach space with norm  $\|\cdot\|$ .

**DEFINITION 2.4.** A function  $\varphi : S \rightarrow X$  is a *step function* if there exists a finite family  $\{M_n\} \subseteq \Sigma$  of pairwise disjoint sets of finite measure and a finite family  $\{e_n\} \subseteq X$  such that

$$\varphi = \sum_n \chi_{M_n} e_n,$$

where  $\chi_{M_n}$  indicates the characteristic function of the set  $M_n$ .

The family of all step functions  $\varphi : S \rightarrow X$  is a real vector space.

DEFINITION 2.5. A function  $f: S \rightarrow X$  is  $X$ -measurable if there is a sequence  $\{\varphi_k\}$  of step functions such that  $\varphi_k \rightarrow f$  in  $X$   $\mu$ -a.e. as  $k \rightarrow \infty$ .

We observe that, by definition, every step function is  $X$ -measurable.

LEMMA 2.6. Let  $\{S_n\}_{n \geq 1} \subseteq \Sigma$  be a countable partition of  $S$  and let  $f: S \rightarrow X$  be a function. Then, the function  $f$  is  $X$ -measurable if and only if for each  $n \geq 1$ , the restriction  $f|_{S_n}: S_n \rightarrow X$  is  $X$ -measurable.

PROPOSITION 2.7. Let  $\{f_n\}$  be a sequence of  $X$ -measurable functions  $f_n: S \rightarrow X$  converging  $\mu$ -a.e. to a function  $f: S \rightarrow X$ . Then the function  $f$  is  $X$ -measurable.

DEFINITION 2.8. A set function  $m: \Sigma \rightarrow X$  is called a *vector valued measure* if

- (1)  $m(\emptyset) = 0$ ,
- (2) for each countable, pairwise disjoint family  $\{A_j\} \subseteq \Sigma$ ,

$$m\left(\bigcup_j A_j\right) = \sum_j m(A_j),$$

where the series is commutatively convergent.

DEFINITION 2.9. The *variation*  $|m|$  of the vector valued measure  $m$  is the set function  $|m|: \Sigma \rightarrow [0, \infty]$  defined for each  $A \in \Sigma$  as

$$|m|(A) = \sup \left\{ \sum_j \|m(A_j)\| \right\},$$

where the supremum is taken over all finite partitions  $\{A_j\} \subseteq \Sigma$  of  $A$ .

LEMMA 2.10. The variation  $|m|$  of a vector valued measure  $m$  is a positive  $\sigma$ -additive measure.

This lemma can be proved by adapting the proof of Theorem 6.2 in [35], p. 117.

LEMMA 2.11. Given two vector valued measures  $m_1, m_2: \Sigma \rightarrow X$ ,

$$|m_1 + m_2| \leq |m_1| + |m_2|.$$

DEFINITION 2.12. Two vector valued measures  $m_1, m_2: \Sigma \rightarrow X$  are *mutually singular*, denoted  $m_1 \perp m_2$ , if the measures  $|m_1|$  and  $|m_2|$  are mutually singular. That is, there is a partition  $S = A \cup B$ ,  $A, B \in \Sigma$ , such that  $|m_1|(A) = 0$  and  $|m_2|(B) = 0$ . In particular, a vector measure  $m: \Sigma \rightarrow X$  and the measure  $\mu$  are mutually singular if the measures  $|m|$  and  $\mu$  are mutually singular.

REMARK 2.13. The following statements are equivalent:

- (1)  $|m|(A) = 0$  for some  $A \in \Sigma$ .
- (2)  $m(A') = 0$  for all  $A' \subseteq A$ ,  $A' \in \Sigma$ .

As a consequence of this remark, we can state the following results:

LEMMA 2.14. *Two vector valued measures  $m_1$  and  $m_2$  are mutually singular if and only if there exists a partition  $S = A \cup B$ ,  $A, B \in \Sigma$ , such that*

$$\begin{aligned} m_1(A') &= 0 \quad \text{for all } A' \subseteq A, A' \in \Sigma, \\ m_2(B') &= 0 \quad \text{for all } B' \subseteq B, B' \in \Sigma. \end{aligned}$$

PROOF. For  $A' \subseteq A$ ,  $A' \in \Sigma$ , by definition,

$$|m_1|(A) \geq ||m_1(A')|| + ||m_1(A - A')||.$$

So  $m_1(A') = 0$ .

Analogously,  $|m_2|(B) = 0$  implies that  $m_2(B') = 0$  for all  $B' \subseteq B$ ,  $B' \in \Sigma$ .  $\square$

LEMMA 2.15. *If two vector valued measures  $m_1, m_2: \Sigma \rightarrow X$  are mutually singular,*

$$|m_1 + m_2| = |m_1| + |m_2|.$$

The proof of Lemma 2.11 and Lemma 2.15 follow closely the proof of Lemma 17 in [4].

DEFINITION 2.16. Given two vector valued measures  $m_1, m_2: \Sigma \rightarrow X$ , we say that  $m_1$  is *absolutely continuous with respect to  $m_2$* , denoted  $m_1 \ll m_2$ , if  $|m_1| \ll |m_2|$ , which means:

If  $A \in \Sigma$  and  $|m_2|(A) = 0$ , then  $|m_1|(A) = 0$  as well. In particular, a vector measure  $m: \Sigma \rightarrow X$  is absolutely continuous with respect to the measure  $\mu$  if  $|m| \ll \mu$ .

REMARK 2.17. According to Remark 2.13,  $m_1 \ll m_2$  if and only if  $A \in \Sigma$  and  $m_2(A') = 0$  for all  $A' \subseteq A$ ,  $A' \in \Sigma$ , implies that  $m_1(A') = 0$  for all  $A' \subseteq A$ ,  $A' \in \Sigma$ .

THEOREM 2.18 (Lebesgue decomposition, [16, p. 189]). *Let  $\mu$  be a positive measure on  $\Sigma$ ,  $X$  a Banach space, and  $m: \Sigma \rightarrow X$  a vector valued measure with  $\sigma$ -finite variation  $|m|$ . If  $|m| + \mu$  has the direct sum property, then  $m$  can be written uniquely in the form of vector valued measures  $m_1, m_2: \Sigma \rightarrow X$  of  $\sigma$ -finite variation such that*

$$m = m_1 + m_2,$$

$$m_1 \ll \mu, m_2 \perp \mu.$$

The  $\sigma$ -finiteness of  $|m|$  is necessary for the validity of Theorem 2.18 even in the case of signed measures, while the measure space  $(S, \Sigma, \mu)$  does not need to be  $\sigma$ -finite.

Before stating the next result, we define a Bochner integrable function. The Bochner integral is the natural generalisation of the familiar Lebesgue integral to the vector-valued setting.

DEFINITION 2.19 (The Bochner integral). Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space. A function  $f: A \rightarrow E$  is  $\mu$ -Bochner integrable if there exists a sequence of  $\mu$ -simple functions  $f_n: A \rightarrow E$  such that the following two conditions are met:

- (1)  $\lim_{n \rightarrow \infty} f_n = f$   $\mu$ -almost everywhere;
- (2)  $\lim_{n \rightarrow \infty} \int_A \|f_n - f\| d\mu = 0$ .

THEOREM 2.20 (Radon-Nikodym). *For a vector valued measure  $m: \Sigma \rightarrow X$ , the following statements are equivalent:*

- (1) *There exists a unique function  $f: S \rightarrow X$ , Bochner integrable, such that  $m = f d\mu$ .*
- (2) *The vector valued measure  $m$  satisfies the following conditions:*
  - (a)  $m \ll \mu$ .
  - (b)  $|m|$  is a finite measure, that is to say  $|m|: \Sigma \rightarrow [0, \infty)$ .
  - (c) *For each  $A \in \Sigma$  with  $0 < \mu(A) < \infty$ , there exists  $E \subseteq A$ ,  $E \in \Sigma$  and a compact set  $K \subseteq X$  not containing zero, such that  $\mu(E) > 0$  and for all  $E' \subseteq E$ ,  $E' \in \Sigma$ , the set  $m(E')$  is contained in the cone generated by  $K$ .*

Please refer to [17] and [32] for the detailed proof of Theorem 2.20.

REMARK 2.21. When the space  $X$  is finite dimensional, condition 2(c) in Theorem 2.20 is satisfied by any vector valued measure  $m: \Sigma \rightarrow X$ . Indeed, we can select

$$K = \{x \in X : \|x\| = 1\}.$$

Then, the cone generated by  $K$ , defined as

$$\{\lambda x : x \in K, \lambda \geq 0\},$$

becomes  $X$ , so condition 2(c) holds. Thus, Theorem 2.20 reduces to the familiar Radon-Nikodym theorem in this case. For a detailed analysis of the conditions involved in Theorem 2.20, see Chapter 5 in [6].

### 3. Extension of real Gleason measure to complex Gleason measure

DEFINITION 3.1. Following F. Riesz and Sz. Nagy ([33, Section 116]), we shall say that an unbounded operator  $T$  and a bounded operator  $B$  are *permutable* (or *commute*) if

$$BT \subset TB.$$

in other words,  $TB$  is an extension of  $BT$ .

DEFINITION 3.2. Let  $H$  be a complex Hilbert space. Suppose that  $T: H \rightarrow H$  is a bounded operator. The *adjoint* of  $T$ , denoted  $T^*$ , is the unique operator  $T^*: H \rightarrow H$  such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$ . A bounded operator  $T: H \rightarrow H$  is *self adjoint* or *hermitian* if  $T = T^*$ .

DEFINITION 3.3. Let  $H$  be a complex Hilbert space and let  $T \in \mathcal{L}(H)$ . Suppose that  $T^*: H \rightarrow H$  is the adjoint of  $T$ .  $T$  is said to be *normal* if  $TT^* = T^*T$ .

Let  $H$  be a Hilbert space,  $\mathcal{A} \subset \mathcal{L}(H)$  a  $C^*$  algebra of bounded normal operators in  $H$  and  $\mathcal{P}$  the set of orthogonal projectors in  $H$ .



DEFINITION 3.4. A *Gleason measure* is a function  $\mu: \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R}$  which is  $\sigma$ -additive on orthogonal families of projections in  $\mathcal{P}(\mathcal{H})$ , i.e. if  $(S_n)_{n \in \mathbb{N}}$  and  $S$  are orthogonal projectors then  $S$  is the strong limit of the series  $S_n$  i.e.

$$(3.1) \quad \mu(S) = \sum_{n \in \mathbb{N}} \mu(S_n).$$

DEFINITION 3.5. Let  $\{|\psi\rangle, |\varphi\rangle\}$  be two states arbitrarily chosen from the Hilbert space  $H$  except that they are neither identical nor orthogonal to each other i.e.  $\langle\psi|\varphi\rangle \neq 0$ . Then

$$(3.2) \quad \mu(S_\psi) = 1, \mu(S_{\psi^\perp}) = 0 \text{ and } \mu(S_\varphi) = 1, \mu(S_{\varphi^\perp}) = 0$$

where  $S_\psi = |\psi\rangle\langle\psi|$ ,  $S_{\psi^\perp} = |\psi^\perp\rangle\langle\psi^\perp|$ ,  $S_\varphi = |\varphi\rangle\langle\varphi|$  and  $S_{\varphi^\perp} = |\varphi^\perp\rangle\langle\varphi^\perp|$ .

DEFINITION 3.6 ([14, Chapter VII, Definition 2.E.1]). Let  $(X, \mathcal{M})$  be a measurable space (i.e.  $\mathcal{M}$  is a  $\sigma$ -algebra of subsets of  $X$ ). A *spectral measure*  $E$  is a mapping  $E: \mathcal{M} \rightarrow \mathcal{P}$  such that

- (1)  $E(U)$  is an orthogonal projector for every  $U \in \mathcal{M}$ .
- (2)  $E(X) = I$ .
- (3) If  $U = U_1 \cap U_2$ , then  $E(U) = E(U_1) \cdot E(U_2)$ . In particular if  $U_1$  and  $U_2$  are disjoint, then  $E(U_1)$  and  $E(U_2)$  are orthogonal.

DEFINITION 3.7. Let  $\mu: \mathcal{P} \rightarrow \mathbb{R}$  be a Gleason measure. Then  $\mu$  is said to be *concentrated on a subspace*  $S_0$  if  $S \subset S_0^\perp$  implies that  $\mu(S) = 0$ . In terms of projections, we can express the same idea by saying that  $\mu$  is concentrated on a projector  $P_0$  if for any projector  $P \in \mathcal{P}(\mathcal{H})$ ,  $P_0 P = 0$  implies that  $\mu(P) = 0$ . We note this by  $\mu \subset S_0$  or  $\mu \subset P_0$ . Furthermore, if the set  $\{P \in \mathcal{P} : \mu(P) = 0\}$  has a greatest element,  $P_0$ , then  $I - P_0$  is called the *strong support* of  $\mu$ . Evidently,  $\mu(P) = 0$  if and only if  $P(I - P_0) = 0$  (see [25]).

DEFINITION 3.8. Let  $\lambda, \alpha: \mathcal{P} \rightarrow \mathbb{R}$  be two Gleason measures. The measure  $\lambda$  is said to be *absolutely continuous with respect to*  $\alpha$  and we write  $\lambda \ll \alpha$ , if  $\alpha(P) = 0$  implies  $\lambda(P) = 0$ . Two Gleason measures  $\lambda$  and  $\alpha$  are said to be *mutually singular* (written  $\lambda \perp \alpha$ ), if there exists an orthogonal decomposition  $I = P_0 + Q_0$  with  $P_0, Q_0$  orthogonal projections such that,  $P_0 Q_0 = Q_0 P_0 = 0$  and  $\lambda \subset P_0, \mu \subset Q_0$ .

DEFINITION 3.9. For  $1 \leq p < \infty$ , we denote by  $L_p$  the class of bounded operators  $(T)$  which satisfy the following condition: for each orthonormal system  $\{\varphi_k, k \in K\}$  in  $H$ ,  $\sum_{k \in K} |\langle T\varphi_k, \varphi_k \rangle|^p < \infty$ .  $L_p$  is a two sided ideal in  $\mathcal{L}(H)$ , thus it is contained in the ideal of compact operators ([34]).

REMARK 3.10. In order to define the trace of an operator  $A$ , we need the series

$$\mathrm{Tr}(T) = \sum_{k \in K} \langle T\varphi_k, \varphi_k \rangle$$

to be absolutely convergent, where  $\mathrm{Tr}$  denotes the trace. So it is natural to define the trace for operators in  $L_1$ . We call the operators in  $L_1$ , operators of trace class. Then if  $A$  is a trace class operator and  $B$  is bounded,  $AB$  is also of trace class. Moreover, we have that

$$|\mathrm{Tr}(AB)| \leq \|B\| \mathrm{Tr}(|A|).$$

We also recall that a frame function of weight  $w$  in  $H$  is a real valued function  $f$  defined on the unit sphere of  $H$  such that if  $\{\varphi_n\}$  is an orthonormal basis of  $H$  then

$$\sum_n f(\varphi_n) = w.$$

We denote  $f_\mu$  the frame function associated to  $\mu$  ([19]).

It is known that a real Gleason's measure  $\mu$  verifying  $|\mu(S)| \leq K$ ,  $S \in \mathcal{S}$  (where  $\mathcal{S}$  is the collection of subspaces in  $H$ ), is represented by a self-adjoint operator ([36]). An analogous result holds for a complex Gleason measure.

In order to extend real Gleason measure to complex Gleason measure, we choose two real maps  $\mu_1, \mu_2: \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R}$  and consider

$$\mu_C(P) = \mu_1(P) + i\mu_2(P)$$

with the imaginary unit  $i$ . Now by Gleason's theorem (see [19]), we have

$$\mu_1(P) = \mathrm{Tr}(AP),$$

and

$$\mu_2(P) = \mathrm{Tr}(BP),$$

where  $P$  is an orthogonal projector and  $A$  and  $B$  are self adjoint operators. Then clearly, the map  $\mu_C$  satisfies (3.1) by its linearity and is written as

$$\mu_C(P) = \mu_1(P) + i\mu_2(P) = \mathrm{Tr}(\rho P), \quad \rho = A + iB,$$

where  $A$  and  $B$  are self adjoint trace class operators associated with  $\mu_1$  and  $\mu_2$  respectively.

**THEOREM 3.11** (Gleason's theorem for complex vector valued measures). *Let  $H$  be a separable Hilbert space (in three or more dimensions), and  $\mu_C : \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{C}$  a complex vector valued measure that satisfies conditions (3.1) and (3.2) for two non-identical states  $|\psi\rangle, |\varphi\rangle$  with  $\langle\varphi, \psi\rangle \neq 0$ . Then*

$$\mu_C(P) = \text{Tr}(\rho P),$$

$$\rho = \alpha \frac{|\psi\rangle\langle\varphi|}{\langle\varphi, \psi\rangle} + (1 - \alpha) \frac{|\varphi\rangle\langle\psi|}{\langle\psi, \varphi\rangle}$$

for some  $\alpha \in \mathbb{C}$ .

Please refer to [29] for the detailed proof of Theorem 3.11.

**THEOREM 3.12.** *Consider  $\rho = A + iB \in L_1$ , where  $A$  and  $B$  are self adjoint operators and  $\rho = |\rho|u$  the polar decomposition of  $\rho$ . Then  $|\rho|$  defines a positive measure and*

$$|\text{Tr}(\rho P)| \leq \text{Tr}(|\rho|P), \quad P \in \mathcal{P}.$$

**PROOF.**  $\text{Tr}(\rho P)$  and  $\text{Tr}(|\rho|P)$  are well defined since  $\rho \in L_1$ . Let  $\rho = u|\rho| = |\rho|u$  with  $|\rho| > 0$  and  $u^* = u^{-1}$ . Let  $a$  be the positive square root of  $|\rho|$ . Consider  $\{e_i\}$  an orthonormal basis of  $H$  and  $P \in \mathcal{P}$ , then

$$\begin{aligned} |\text{Tr}(\rho P)| &= |\text{Tr}(ua^2P^2)| = |\text{Tr}(Pa u a P)| \\ &\leq \sum_i |\langle u a P e_i, a P e_i \rangle| \leq \sum_i \|a P e_i\|^2 = \text{Tr}(|\rho|P). \end{aligned} \quad \square$$

**REMARK 3.13.** From Theorem 3.12, a bounded complex Gleason measure  $\mu_C$  is bounded by a positive measure. In this case, if  $\mu_C$  is real then the following decomposition is obtained

$$\mu_C = \mu_C^+ - \mu_C^-$$

with  $\mu_C^+ = (\mu_{c\|\rho\|} + \mu_C)/2$  and  $\mu_C^- = (\mu_{c\|\rho\|} - \mu_C)/2$ .

#### 4. Integral with respect to a complex Gleason measure

The notion of integral of an operator with respect to a Gleason measure is required. To motivate this notion, we present some results from De Nápoli and Mariani ([15, Section 3]) that works for complex Gleason measures.

Consider a self-adjoint operator that is a finite linear combination of projections:

$$A = \sum_{i=1}^n \lambda_i P_i,$$

where  $P_i \in \mathcal{P}$  and  $P_i P_j = 0$  if  $i \neq j$ .

In analogy with standard measure theory, call these operators simple operators. Then it is natural to define the integral of a simple operator with respect to  $\mu$  by:

$$\int A d\mu = \sum_{i=1}^n \lambda_i \mu(P_i).$$

We shall extend this notion of integral to the class of self-adjoint bounded operators. To do that refer to the spectral theorem, in the following formulation:

**THEOREM 4.1.** *To each (possibly unbounded) self-adjoint operator  $A$  in a Hilbert space  $H$  corresponds a spectral measure  $E = E_A$  (defined on the Borel sets of  $\mathbb{R}$ ) such that:*

(1)

$$A = \int_{-\infty}^{\infty} \lambda dE$$

*in the sense that*

$$Ax = \lim_{n \rightarrow \infty} \int_{-n}^n \lambda E(\lambda)x$$

*and*

$$D(A) = \left\{ x \in H : \int_{-\infty}^{\infty} \lambda^2 \langle E(d\lambda)x, x \rangle < \infty \right\}.$$

- (2) For each Borel set  $U \subset \mathbb{R}$ ,  $E(U)$  commutes with any bounded operator that commutes with  $A$ , and

$$E(U)A = \int_U \lambda dE.$$

- (3) For any real Borel measurable function  $f(\lambda)$  we have

$$f(A) = \int_{-\infty}^{\infty} f(\lambda) dE$$

with

$$D(f(A)) = \{x \in H : \int_{-\infty}^{\infty} |f(\lambda)|^2 \langle E(d\lambda)x, x \rangle < \infty\}.$$

- (4) The spectral measure  $E$  is supported in the spectrum  $\sigma(A)$  of  $A$ , i.e., for every Borel set  $U \subset \mathbb{R}$ ,  $E(U) = E(U \cap \sigma(A))$ .

PROOF. Please refer to [14] for the proof of Theorem 4.1.  $\square$

In the case of a simple operator, the spectral measure  $E_A$  is given by:

$$E_A(U) = \sum_{i: \lambda_i \in U} P_i.$$

Consider the measure  $\mu \circ E_A$  on the Borel sets of  $\mathbb{R}$ , then if  $A$  is a simple operator,

$$(4.1) \quad \int A d\mu = \int_{-\infty}^{\infty} \lambda d(\mu \circ E_A).$$

So for any self-adjoint operator  $A$  define  $\int A d\mu$  using equation (4.1). In a similar way, using more general versions of the spectral theorem, it is possible to define the integral  $\int A d\mu$  when  $A$  is a normal operator. Please refer to [14] for more details.

REMARK 4.2. Following F. Riesz and Sz. Nagy ([33, Section 130]), if  $A$  and  $B$  are two permutable self-adjoint operators, it follows that  $A$  and  $B$  are permutable if  $E_A(U)$  and  $E_B(V)$  are permutable for any measurable sets  $U, V \subset \mathbb{R}$ , with  $E_A, E_B$  the spectral measures associated to  $A, B$ . In that case

there exists a spectral measure  $E_{A,B}$  defined for the Borel subsets of  $\mathbb{R}^2$ , such that

$$E_{A,B}(U \times V) = E_A(U)E_B(V)$$

for all “measurable rectangles”. Furthermore,

$$\begin{aligned} A &= \int \int_{\mathbb{R}^2} \lambda_1 dE_{A,B}(\lambda_1, \lambda_2), \\ B &= \int \int_{\mathbb{R}^2} \lambda_2 dE_{A,B}(\lambda_1, \lambda_2). \end{aligned}$$

PROPOSITION 4.3. *If  $A$  and  $B$  are permutable self-adjoint operators then*

$$(4.2) \quad \int (A + B) d\mu = \int A d\mu + \int B d\mu.$$

PROOF. Consult [15, Section 3] for the proof.  $\square$

REMARK 4.4. This property does not hold if  $A$  and  $B$  are not permutable, as can be seen from the following example: We consider the Hilbert space  $H = \mathbb{R}^2$ , and denote by  $S_\theta$  the 1-dimensional subspace generated by the vector  $(\cos \theta, \sin \theta)$ . Given a function  $f: [0, \pi/2) \rightarrow [0, 1]$  we can define a Gleason measure  $\mu$  in  $H$  by

$$\mu(S_\theta) = \begin{cases} f(\theta) & \text{if } 0 \leq \theta < \pi/2, \\ 1 - f(\theta - \frac{\pi}{2}) & \text{if } \pi/2 \leq \theta < \pi, \end{cases}$$

and  $\mu(0) = 0$ ,  $\mu(H) = 1$ . If we take  $A$  to be the projection onto  $S_0$  and  $B$  to be the projection onto  $S_{\pi/4}$ , it can be easily seen that (4.2) does not hold for  $f$  in general.

REMARK 4.5. Let  $(X, \mathcal{M})$  be a measurable space. If  $E: \mathcal{M} \rightarrow P(H)$  is a spectral measure and

$$A = \int_X \lambda dE,$$

it follows that

$$\langle Ax, y \rangle = \int \lambda dE[x, y],$$

where

$$E[x, y](U) = \langle E(U)x, y \rangle.$$

PROPOSITION 4.6. *Let  $\mu: \mathcal{S} \rightarrow \mathbb{R}$  be a Gleason measure, and assume that  $\mu$  is represented by the trace-class operator  $\rho$ , i.e.*

$$\mu(S) = \text{Tr}(\rho P_S) \quad \forall S \in \mathcal{S}.$$

*Then for any (not necessarily bounded)  $\mu$ -integrable self-adjoint operator  $A$  in  $H$  we have that*

$$\int A d\mu = \text{Tr}(\rho A).$$

PROOF. Consult [15, Section 3] for the proof.  $\square$

In order to justify these formal computations, we may assume first that  $A$  is a positive operator, and then for the general case, we use the decomposition  $A = A^+ - A^-$ .

REMARK 4.7. It follows that when the Gleason measure  $\mu$  is represented by a trace-class operator, the linearity property (4.2) holds for any operators  $A, B$  (even if they are not permutable). Please see [15] for more details.

LEMMA 4.8. *Let  $\mu$  be a finite non-negative Gleason measure and  $A$  a bounded self-adjoint operator. If*

$$m = m(A) = \inf_{\|x\|=1} \langle Ax, x \rangle,$$

$$M = M(A) = \sup_{\|x\|=1} \langle Ax, x \rangle$$

*are the lower and upper bounds of  $A$  respectively, then the following result is obtained:*

$$m(A)\mu(I) \leq \int A d\mu \leq M(A)\mu(I).$$

*In particular,*

$$\left| \int A d\mu \right| \leq \|A\| \mu(I).$$

PROOF. Consult [15, Section 3] for the proof.  $\square$

DEFINITION 4.9. Let  $\mu$  be a (non-negative) Gleason measure and  $A$  a self-adjoint operator. We say that  $A = 0$  *a.e. with respect to  $\mu$*  if  $\mu(\text{Ker}(A)^\perp) = 0$ .

LEMMA 4.10. *If  $A = 0$  a.e. with respect to  $\mu$ , then  $\int A d\mu = 0$ .*

PROOF. Consult [15, Section 3] for the proof.  $\square$

The generalization of all these results for complex Gleason measure can be done as follows. Consider operators  $A_1$  and  $A_2$ . Then all the above results can be written for both  $A_1$  and  $A_2$ , furthermore it can be considered, without loss of generality, that both  $A_1$  and  $A_2$  are positive. Next consider the operator  $A = A_1 + iA_2$ . Then all the above results hold for  $A$ .

## 5. Lebesgue decomposition with respect to a representable measure

In this section, we present a different approach for obtaining a Lebesgue decomposition for complex Gleason measures, that applies when the measure  $\mu$  is a representable measure. Following [15, Section 5] we get:

THEOREM 5.1. *Let  $\mu, \lambda$  be two Gleason measures defined on a Hilbert space  $H$  and assume that  $\mu$  is represented by a positive trace class operator  $\rho_1$ . Then, there exist two Gleason measures  $\lambda_a$  and  $\lambda_s$  such that*

$$\lambda(P) = \lambda_a(P) + \lambda_s(P)$$

*for any projector that commutes with the projector  $P_R$  onto the range  $R(\rho_1)$  of  $\rho_1$ ,  $\lambda_a \perp \lambda_s$ ,  $\lambda_a \ll \mu$  and  $\lambda_s$  is singular with respect to  $\mu$ . Moreover, if  $\lambda$  is also a representable measure, this decomposition holds for any  $P \in \mathcal{P}(\mathcal{H})$ .*

PROOF. Consult [15, Section 5] for the proof.  $\square$

## 6. A version of the Radon-Nikodym theorem for complex Gleason measures

We will proceed in a similar way as in [15], but first we will present some results from [15]. Let  $A$  be a normal operator,  $\mu$  a positive Gleason measure and define

$$\lambda_A(S) = \int A|_S d\mu = \int_S A d\mu$$



where  $A|_S$  is the operator  $AP_S$ , then  $\lambda_A$  is a Gleason measure on the set of  $A$ -invariant subspaces (with the identification of  $S$  with  $P_S$  we may think it as the set of projectors such that  $P_SA \subset AP_S$ ). In fact, if  $S = \bigoplus_{n \in \mathbb{N}} S_n$ , then

$$P_S = \sum_{n \in \mathbb{N}} P_{S_n}$$

and using Proposition 4.3 we see that

$$\int_S A \, d\mu = \sum_{n \in \mathbb{N}} \int_{S_n} A \, d\mu;$$

$AP_{S_i}$  and  $AP_{S_j}$  commute, because  $S_i, S_j$  are  $A$ -invariant subspaces.

We remark that in the special case in which  $\mu$  is a Gleason measure represented by a trace-class operator, we consider  $\lambda_A$  to be defined for all closed subspaces of  $H$ , since as observed before, in that case the linearity property (4.2) holds without restrictions.

LEMMA 6.1.  $\lambda_A$  is absolutely continuous with respect to  $\mu$ .

PROOF. Refer to [15, Section 6] for the complete proof.  $\square$

Now suppose that we are given two Gleason measures  $\lambda, \mu$  such that  $\lambda$  is absolutely continuous with respect to  $\mu$ . It is natural to ask if  $\lambda = \lambda_A$  for some self-adjoint operator  $A$ . Recall the following results in [15].

LEMMA 6.2. *Let us assume that  $\lambda$  and  $\mu$  are Gleason measures represented by the trace-class operators  $\rho_1$  and  $\rho_2$  respectively, then  $\lambda \ll \mu$  if and only if  $\text{Ker}(\rho_2) \subset \text{Ker}(\rho_1)$*

PROOF. Refer to [15] for the complete proof.  $\square$

THEOREM 6.3 (Radon-Nikodym theorem for complex Gleason measures). *Let  $\lambda, \mu$  be two positive representable Gleason measures, and  $\rho_1, \rho_2$  be their respective positive density operators (see [28], [38]), so that*

$$\lambda(S) = \text{Tr}(\rho_1 P_S),$$

$$\mu(S) = \text{Tr}(\rho_2 P_S).$$

Assume that  $\lambda \ll \mu$ . Then there exists a (non necessarily bounded) self-adjoint operator  $A$  such that

$$\lambda(T) = \int AP_T d\mu$$

for any closed subspace  $T$  of  $H$ .

PROOF. For a complete proof of Theorem 6.3 please refer to [15].  $\square$

REMARK 6.4. Under the assumptions of Lemma 6.2,  $\lambda$  and  $\mu$  are positive Gleason measures. A similar result holds if  $\lambda$  is assumed to be a complex Gleason measure (let  $\lambda = \psi + i\nu$  where  $\psi$  and  $\nu$  are positive representable Gleason measures), represented by an operator  $\rho_2$ . In that case, let  $B$  and  $C$  be self-adjoint operators such that  $\psi(T) = \int BP_T d\mu$  and  $\nu(T) = \int CP_T d\mu$ , following the same proof as Theorem 6.3 in [15] we obtain:

$$\lambda(T) = \psi(T) + i\nu(T) = \int BP_T d\mu + i \int CP_T d\mu = \int (B + iC)P_T d\mu.$$

REMARK 6.5. Gleason theorem can be seen as a version of the Radon-Nikodym theorem.

Consider the Gleason measure  $\Delta$  given by

$$\Delta(S) = \dim(S).$$

It is clear that  $\Delta$  is a non-negative Gleason measure (though it may take the value  $+\infty$ ). Then if  $A = \sum \lambda_i P_{S_i}$  is a simple self-adjoint operator,

$$\int A d\Delta = \sum \lambda_i \dim(S_i) = \text{Tr}(A).$$

This identity holds also for any operator  $A$  of trace class (i.e.,  $\Delta$ -integrable). If  $\mu$  is another Gleason measure, it is clear that  $\mu$  is absolutely continuous with respect to  $\Delta$  since  $\mu(\{0\}) = 0$ . Gleason theorem says that there exists a self-adjoint operator  $\rho$  such that

$$\mu(S) = \int \rho P_S d\Delta$$

for any closed subspace  $S$ . The condition that  $H$  should be separable means that  $\Delta$  should be a  $\sigma$ -finite Gleason measure (hypothesis of the usual Radon-Nikodym theorem). Furthermore as done previously this result can be extended to a vector-valued measure and, in particular, to a complex Gleason

measure. For this particular case it would be enough to represent the complex Gleason measure in binomial form with two non-negative real Gleason measures and continue in a similar fashion as in Remark 6.5 for each measure, then this would yield an operator  $\rho$ .

A Gleason's measure on a family of commuting orthogonal projections behaves as an ordinary measure on  $\sigma$ -algebras, in the sense that it is possible to define a variation.

## 7. Vector valued Nemytsky operator

We first present some results and definitions from [3]. We take  $(S, \Sigma, \mu)$  to be a complete  $\sigma$ -finite measure space. Suppose that  $X$  and  $Y$  are real Banach spaces with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , respectively.

DEFINITION 7.1. A function  $g: S \times X \rightarrow Y$  is called a *vector valued  $N$ -function* if it satisfies the conditions:

- (1) For each  $u \in X$ , the function  $x \rightarrow g(x, u)$  is  $Y$ -measurable.
- (2) There is a  $\mu$ -null set  $A$  such that for each  $x \in S \setminus A$ , the function  $u \rightarrow g(x, u)$  is continuous.

It is therefore clear that,  $B^p(X)$  is the space of  $X$ -measurable functions  $f: S \rightarrow X$  for which the function  $x \rightarrow \|f(x)\|_X \in L^p$ .

In the scalar case, conditions (1) and (2) in Definition 7.1 are referred to as the Caratheodory's conditions (see [22, p. 20] and the references therein). For example,

$$(7.1) \quad g(x, u) = \sum_{i=1}^n a_i(x) \|u - b_i(x)\|_X + T(u)$$

is a vector valued  $N$ -function when  $a_i \in B^\infty(Y)$ ,  $b_i \in B^1(X)$  and  $T \in L(X, Y)$ , the space of linear and continuous operators from  $X$  into  $Y$ . We call the  $N$ -function given by (7.1) a vector valued piecewise linear  $N$ -function, by analogy with the case of scalar piecewise linear  $N$ -functions (see [4, Definition 18]).

Let

$$\mathcal{M}_X = \left\{ \begin{array}{l} m: \Sigma \rightarrow X : \text{ vector valued measure satisfying} \\ \text{conditions 2(a), 2(b) and 2(c) in Theorem 2.20} \end{array} \right\}.$$

The map  $\Lambda: B^1(X) \rightarrow \mathcal{M}_X$  defined as  $\Lambda(f) = fd\mu$  is an isomorphism of real vector spaces. Moreover we have the following result:

PROPOSITION 7.2 ([16, p. 174, Proposition 10]). *For  $f \in B^1(X)$ ,*

$$|fd\mu| = \|f\| d\mu,$$

*where  $\|f\|$  denotes the scalar function  $x \rightarrow \|f(x)\|_X$ . As a consequence,  $\mathcal{M}_X$ , with the variation norm  $\|m\| = |m|(S)$ , is a Banach space and  $\Lambda$  becomes an isometric isomorphism.*

PROPOSITION 7.3. *Given an  $N$ -function  $g: S \times X \rightarrow Y$  and given an  $X$ -measurable function  $f: S \rightarrow X$ , the function  $x \rightarrow g(x, f(x))$  from  $S$  into  $Y$  is  $Y$ -measurable.*

PROOF. For a proof of the result see [3]. □

DEFINITION 7.4. Given a vector valued  $N$ -function  $g$ , the *Nemytsky operator*  $N_g(f)$  is defined for an  $X$ -measurable function  $f$  as

$$(7.2) \quad N_g(f)(x) = g(x, f(x)).$$

It is clear that Proposition 7.3 implies that the Nemytsky operator  $N_g$  maps  $X$ -measurable functions to  $Y$ -measurable functions.

REMARK 7.5. It is known ([37, p. 155], [22, p. 20]), that (7.2) defines in the scalar case a continuous and bounded operator from  $L^1$  into itself if and only if there exist a function  $a \in L^1$  and a number  $b \geq 0$  such that

$$(7.3) \quad |g(x, u)| \leq a(x) + b|u|$$

$\mu$ -a.e. in  $S$ . A similar proof of this result can be established for the vector valued case, if we write (7.3) in the form

$$(7.4) \quad \|g(x, u)\|_Y \leq a(x) + b\|u\|_X$$

$\mu$ -a.e. in  $S$ , with  $a \in L^1$  and  $b \geq 0$ . Thus, for an  $N$ -function  $g$  satisfying (7.4), the Nemytsky operator  $N_g$  is a bounded and continuous operator from  $B^1(X)$  into  $B^1(Y)$ .

PROPOSITION 7.6. *If  $g$  is a vector valued  $N$ -function satisfying (7.4), there exists a unique operator  $\bar{N}_g: \mathcal{M}_X \rightarrow \mathcal{M}_Y$  such that the diagram*

$$\begin{array}{ccc}
B^1(X) & \xrightarrow{N_g} & B^1(Y) \\
\Lambda \downarrow & & \downarrow \Lambda \\
\mathcal{M}_X & \xrightarrow{\overline{N}_g} & \mathcal{M}_Y
\end{array}$$

is commutative. That is to say,

$$\Lambda \circ N_g = \overline{N}_g \circ \Lambda$$

on  $B^1(X)$ .

PROOF. As in the scalar case (see [4, Proposition 15]), we propose

$$(7.5) \quad \overline{N}_g(f d\mu) = g(\cdot, f(\cdot)) d\mu.$$

The Radon-Nikodym theorem determines the function  $f$  in (7.5)  $\mu$ -almost everywhere. However, if  $h = f$   $\mu$ -a.e., there is a  $\mu$ -null set  $A$  such that  $h(x) = f(x)$  for  $x \in S \setminus A$ . Then, for any  $E \in \Sigma$ ,

$$\begin{aligned}
\overline{N}_g(f d\mu)(E) &= \int_E g(\cdot, f(\cdot)) d\mu \\
&= \int_{(S \setminus A) \cap E} g(\cdot, h(\cdot)) d\mu = \overline{N}_g(h d\mu)(E).
\end{aligned}$$

This observation and the properties of the Nemytsky operator  $N_g$ , imply that the operator  $\overline{N}_g$  is well defined. Moreover,

$$\begin{aligned}
\Lambda \circ N_g(f) &= g(\cdot, f(\cdot)) d\mu \\
&= \overline{N}_g(f d\mu) = \overline{N}_g \circ \Lambda(f),
\end{aligned}$$

for every  $f \in B^1(X)$ . Finally, the operator  $\overline{N}_g$  is unique, for if there is another operator, say  $\tilde{N}_g: \mathcal{M}_X \rightarrow \mathcal{M}_Y$ , that also makes the diagram commutative, we will have

$$\begin{aligned}
\tilde{N}_g(f d\mu) &= \tilde{N}_g \circ \Lambda(f) \\
&= \Lambda \circ N_g(f) = \overline{N}_g \circ \Lambda(f) \\
&= \overline{N}_g(f d\mu),
\end{aligned}$$

for every  $f \in B^1(X)$ , or  $\tilde{N}_g = \overline{N}_g$  on  $\mathcal{M}_X$ . This completes the proof of the proposition.  $\square$

With all these results we now define the map  $g \rightarrow \overline{N}_g$  as a functional calculus. We denote  $\mathcal{N}$  the family of vector valued  $N$ -functions  $g: S \times X \rightarrow X$  satisfying the condition

$$(7.6) \quad \|g(x, u)\|_X \leq a(x) + b\|u\|_X$$

$\mu$ -a.e. in  $S$ , with  $a \in L^1$  and  $b \geq 0$ . We also define

$$\overline{\mathcal{N}} = \{\overline{N}_g : g \in \mathcal{N}\}.$$

PROPOSITION 7.7. (1) *The spaces  $\mathcal{N}$  and  $\overline{\mathcal{N}}$  are real vector spaces and the map  $g \rightarrow \overline{N}_g$  from  $\mathcal{N}$  into  $\overline{\mathcal{N}}$  is linear.*

(2)  *$B^1(X) \subseteq \mathcal{N}$ , in the sense that every function  $g \in B^1(X)$  defines an  $N$ -function  $g(x)$  that satisfies (7.6). Moreover, given  $g \in B^1(X)$ ,*

$$\overline{N}_g(f d\mu) = g d\mu.$$

(3) *The space  $\mathcal{N}$  is closed under the composition operation*

$$(g_1 \circ g_2)(x, u) = g_1(x, g_2(x, u))$$

and

$$(7.7) \quad \overline{N}_{g_1 \circ g_2} = \overline{N}_{g_1} \circ \overline{N}_{g_2}.$$

PROOF. Clearly the spaces  $\mathcal{N}$  and  $\overline{\mathcal{N}}$  are real vector spaces. Moreover, if  $\alpha, \beta \in \mathbb{R}$  and  $g_1, g_2 \in \mathcal{N}$ ,

$$\begin{aligned} \overline{N}_{\alpha g_1 + \beta g_2}(f d\mu) &= [\alpha g_1(\cdot, f(\cdot)) + \beta g_2(\cdot, f(\cdot))] d\mu \\ &= \alpha \overline{N}_{g_1}(f d\mu) + \beta \overline{N}_{g_2}(f d\mu). \end{aligned}$$

It is clear that the Bochner integrable functions define  $N$ -functions in  $\mathcal{N}$ . Moreover, given  $g \in B^1(X)$  and  $f \in B^1(X)$ ,

$$N_g(f) = g$$

and

$$\overline{N}_g(f d\mu) = g d\mu,$$

according to Proposition 7.6.

To see that the space  $\mathcal{N}$  is closed under the composition operation  $\circ$ , we observe that there exist  $\mu$ -null sets  $A_1$  and  $A_2$  such that for each  $x \in S \setminus A_i$ ,

the map  $u \rightarrow g_i(x, u)$  is continuous from  $X$  into itself, for  $i = 1, 2$ . Thus, if  $x \in S \setminus (A_1 \cup A_2)$ , the map  $u \rightarrow g_1(x, g_2(x, u))$  is continuous as well. We now fix  $u \in X$ . Then, the function  $x \rightarrow g_i(x, u)$  from  $S$  into  $X$  is  $X$ -measurable, so Proposition 7.3 implies that the function  $x \rightarrow g_1(x, g_2(x, u))$  is also  $X$ -measurable. Furthermore,

$$\begin{aligned} \|g_1(x, g_2(x, u))\|_X &\leq a_1(x) + b_1 \|g_2(x, u)\|_X \\ &\leq a_1(x) + b_1(a_2(x) + b_2 \|u\|_X) \\ &= a_1(x) + b_1 a_2(x) + b_1 b_2 \|u\|_X. \end{aligned}$$

So,  $g_1 \circ g_2 \in \mathcal{N}$ . Finally, we prove (7.7).

$$\begin{aligned} \overline{N}_{g_1 \circ g_2}(f d\mu) &= g_1(\cdot, g_2(\cdot, f(\cdot))) d\mu \\ &= \overline{N}_{g_1}[\overline{N}_{g_2}(f d\mu)] \\ &= (\overline{N}_{g_1} \circ \overline{N}_{g_2})(f d\mu). \end{aligned}$$

This completes the proof of the proposition.  $\square$

## 8. The operator $\overline{N}_g$ for vector valued piecewise linear $N$ -functions

In this section we present some results in [3] on how to extend to vector valued measures the Nemytsky operator  $N_g$  associated to the piecewise linear  $N$ -function  $g$  defined by (7.1). We begin with the following

**LEMMA 8.1.** *If  $m_1, m_2: \Sigma \rightarrow X$  are vector valued measures and  $m_1 \perp \mu, m_2 \perp \mu$ , then  $m_1 + m_2 \perp \mu$ .*

**PROOF.** The following proof was extracted from [3] section 5. According to Remark 2.13 and Lemma 2.14, there exist partitions  $S = X \cup Y = V \cup W$ , where  $X, Y, V, W \in \Sigma$ , such that

$$\begin{aligned} \mu(Y) &= \mu(W) = 0, \\ m_1(X') &= 0, \quad \text{for all } Y' \subseteq Y, Y' \in \Sigma, \\ m_2(V') &= 0, \quad \text{for all } V' \subseteq V, V' \in \Sigma. \end{aligned}$$

We consider now the partition  $S = (X \cap V) \cup (Y \cup W)$ . For this partition,

$$\mu(Y \cup W) \leq \mu(Y) + \mu(W) = 0.$$

If we fix  $R \subseteq X \cap V, R \in \Sigma$ ,

$$(m_1 + m_2)(R) = m_1(R) + m_2(R) = 0,$$

proving that the measures  $m_1 + m_2$  and  $\mu$  are mutually singular. This completes the proof of the lemma.  $\square$

The following result identifies the Lebesgue decomposition of  $\overline{N}_g(m)$ :

**PROPOSITION 8.2.** *If  $m \in \mathcal{F}_X$  and  $m = fd\mu + m_s$  is the Lebesgue decomposition of  $m$ , then*

$$\overline{N}_g(m) = \overline{N}_g(fd\mu) + \left( \sum_{i=1}^n a_i \right) |m_s| + T \circ m_s,$$

with  $\left( \sum_{i=1}^n a_i \right) |m_s| + T(m_s) \perp \overline{N}_g(fd\mu)$ .

**PROOF.** Please refer to [3, Section 5] for the proof.  $\square$

## 9. Main results

In this section we first recall some results from [3, Section 6], then present our main result; Theorem 9.7.

**LEMMA 9.1.** *Let  $G: [0, T] \times S \times X \rightarrow X$  be a function satisfying the conditions:*

- (1)  $\|G(t, x, u)\|_X \leq a(x) + b\|u\|_X$ , for some  $a \in L^1$  and  $b \geq 0$ , for  $\mu$ -a.e.  $x \in S$ ,  $0 \leq t \leq T$  and  $u \in X$ .
- (2) The function  $x \rightarrow G(t, x, u)$  is  $\Sigma$ -measurable for each  $0 \leq t \leq T$  and  $u \in X$ .
- (3) There exists  $C > 0$  such that  $\|G(t, x, u_1) - G(t, x, u_2)\|_X \leq C\|u_1 - u_2\|_X$ , for  $0 \leq t \leq T$ ,  $u_1, u_2 \in X$ , and  $\mu$ -a.e.  $x \in S$ .



(4) *There exists  $C > 0$  such that*

$$\|G(t_1, x, u) - G(t_2, x, u)\|_X \leq C \|u\|_X |t_1 - t_2|,$$

*for  $0 \leq t_1, t_2 \leq T$ ,  $u \in X$ , and  $\mu$ -a.e.  $x \in S$ .*

*Then, the following properties hold:*

- a) *For each  $0 \leq t \leq T$ , the function  $G_t: S \times X \rightarrow X$  defined as  $G_t(x, u) = G(t, x, u)$  is an  $N$ -function.*
- b) *For each  $0 \leq t \leq T$ , the Nemystky operator  $N_{G_t}$  maps  $B^1(X)$  to itself.*
- c) *The function  $f(t, x) \rightarrow N_{G_t}(f(t, \cdot))(x)$  maps  $C[0, T; B^1(X)]$  continuously into itself.*

PROOF. The following proof was extracted from [3] and closely follows the proof of Lemma 21 in [4]. The proof of property a) is a direct application of conditions 2 and 3 of Lemma 9.1, while, in particular, the proof of b) follows from condition 1 of Lemma 9.1 and directly from Remark 7.5. For the proof of property c) we observe that by substituting  $L^1$  for  $B^1(X)$  and modifying the norm to the corresponding space the proof of Lemma 21 in [4] holds and proceeds in a similar manner to obtain the desired result.  $\square$

If the function  $G: [0, T] \times S \times X \rightarrow X$  satisfies the hypotheses of Lemma 9.1 and  $m \in C[0, T; \mathcal{M}_X]$ , we define

$$(9.1) \quad \mathcal{A}(m)(t) = \overline{N}_{G_t}(m(t)).$$

Before we proceed we state an important result.

**THEOREM 9.2** (Banach fixed point theorem). *Let  $(M, d)$  be a complete metric space, then each contraction map  $f: M \rightarrow M$  has a unique fixed point.*

PROOF. Please refer to [24, Section 2.3] for the proof.  $\square$

The next result is a well known extension of the Banach fixed point theorem:

**PROPOSITION 9.3.** *Let  $(M, d)$  be a complete metric space and consider a map  $f: M \rightarrow M$ . If there exists  $k \in \{1, 2, \dots\}$  such that the composite map  $f^{(k)}$  is a contraction, then the map  $f$  has a unique fixed point.*

PROOF. By Theorem 9.2,  $f^{(k)}$  has a unique fixed point, say  $m \in M$  with  $f^{(k)}(m) = m$ . Since

$$f^{(k+1)}(m) = f(f^{(k)}(m)) = f(m),$$

it follows that  $f(m)$  is a fixed point of  $f^{(k)}$ , and thus, by the uniqueness of  $m$ , we have  $f(m) = m$ , that is,  $f$  has a fixed point. Since the fixed point of  $f$  is necessarily a fixed point of  $f^{(k)}$ , it implies that  $m$  is unique.  $\square$

**THEOREM 9.4.** *If we assume that the operator  $\mathcal{A}$  is given by (9.1) and the function  $G$  satisfies the conditions stated in Lemma 9.1, the initial value problem*

$$(9.2) \quad \begin{cases} \frac{dm}{dt} + \mathcal{A}(m)(t) &= 0 & \text{for } 0 < t < T, \\ m(0) &= m_0, \end{cases}$$

*will have one and only one solution in  $C^1[0, T; \mathcal{M}_X]$  for each  $m_0 \in \mathcal{M}_X$ .*

**PROOF.** We cite the proof included in [3], this is only a sketch of the proof since it follows closely the proof of Theorem 23 in [4]. We observe that the initial value problem (9.2) has the same solutions in  $C^1[0, T; \mathcal{M}_X]$  as the integral equation

$$(9.3) \quad m(t) = m_0 + \int_0^t \mathcal{A}(m)(s) ds.$$

To prove that (9.3) has one and only one solution in  $C^1[0, T; \mathcal{M}_X]$  it suffices to show that the operator  $\mathcal{T}$  defined on  $C[0, T; \mathcal{M}_X]$  as:

$$\mathcal{T}(m) = m_0 + \int_0^t \mathcal{A}(m)(s) ds$$

has a unique fixed point. According to Proposition 9.3, the operator  $\mathcal{T}$  has a unique fixed point if  $\mathcal{T}^{(k)}$  is a contraction in  $C[0, T; \mathcal{M}_X]$  for some  $k \in \{1, 2, \dots\}$ . The operator  $\mathcal{T}^{(k)}$  will be a contraction in  $C[0, T; \mathcal{M}_X]$  for some  $k \in \{1, 2, \dots\}$  if

$$(9.4) \quad \|\mathcal{T}^{(k)}(m_1) - \mathcal{T}^{(k)}(m_2)\| \leq \frac{C^k T^k}{k!} \|m_1 - m_2\|,$$

for  $m_1, m_2 \in C[0, T; \mathcal{M}_X]$  and  $k \in \{1, 2, \dots\}$ . The estimate (9.4) can be proved by induction, completing the proof of Theorem 9.4.  $\square$

**REMARK 9.5.** We observe that  $C[0, T; \mathcal{M}_b]$  is isometrically isomorphic to  $C[0, T; B^1]$  endowed with the norm

$$\|f\| = \sup_{0 \leq t \leq T} \|f(t)\|_{B^1}.$$

Indeed,  $\|(fd\mu)(t)\|_{\mathcal{M}_b} = \|f(t)\|_{B^1}$ , for each  $0 \leq t \leq T$ .

It follows from Lemma 9.1 and Remark 9.5 that the operator  $\mathcal{A}$  is continuous from  $C[0, T; \mathcal{M}_b]$  into itself.

Next, we present the definition of a positive-operator valued measure before moving on to discuss our main result. We recall that  $(S, \Sigma, \mu)$  is a complete  $\sigma$ -finite measure space and  $X$  is a real Banach space with norm  $\|\cdot\|$ .

**DEFINITION 9.6.** A *positive-operator valued measure* (POVM) is a mapping  $F$  whose values are bounded non-negative self-adjoint operators on a Hilbert space  $H$ , that is,  $F: \Sigma \rightarrow \mathcal{B}(H)_+$  such that:

- (1)  $F(\emptyset) = 0$ ,  $F(S) = I_H$ ; and
- (2)  $F(\bigcup_{i=1}^{\infty} M_i) = \sum_{i=1}^{\infty} F(M_i)$  whenever  $M_i \cap M_j = \emptyset$  for  $i \neq j$ .

In short such an  $F$  is a non-negative countably additive measure on the  $\sigma$ -algebra  $\Sigma$ .

In the following results, we construct a complex Gleason measure that will work as a positive-operator valued measure.

**THEOREM 9.7.** *A complex Gleason measure can be used as an operator measure for the Nemytsky operator.*

**PROOF.** Even though the proof follows from Theorems 6.3 and 9.4, we will provide a detailed proof. We remark that a measure can be used as an operator measure for the Nemytsky operator if given the Nemytsky operator with conditions as in Lemma 9.1, one can construct a complex Gleason measure that will work as a positive-operator valued measure. Consider the initial value problem given in (9.2). We know from Theorem 9.4 that (9.2) has a unique solution,  $m \in C^1[0, T; \mathcal{M}_X]$  (where  $C^1[0, T; \mathcal{M}_X]$  is the set of vector-valued measures continuous in the interval  $[0, T]$  satisfying properties stated in Theorem 2.20). Following Theorem 6.3, we let  $m$  be a representable complex Gleason measure and let  $\rho_1$  be its density operator, so that:

$$m(S) = \text{Tr}(\rho_1 P_S)$$

with  $\rho_1$  positive. Let  $P_S = \{P_i; \text{ where } P_i \text{ is a projector such that } P_i \mathcal{A} \subset \mathcal{A}P_i\}$ . Next we define  $n_{\mathcal{A}}: P_S \rightarrow \mathbb{C}$  by:

$$n_{\mathcal{A}}(S) = \int \mathcal{A}P_S dm,$$

then by Theorem 6.3,  $n_{\mathcal{A}}$  is a complex Gleason measure in the space of  $\mathcal{A}$ -invariant subspaces, thus we can write the complex Gleason measure

$m = \mu + i\nu$ , where  $\mu, \nu$  are positive Gleason measures and obtain

$$n_{\mathcal{A}}(S) = \lambda_{\mathcal{A}}(S) + i\sigma_{\mathcal{A}}(S) = \int \mathcal{A}P_S d\mu + i \int \mathcal{A}P_S d\nu,$$

where  $\lambda_{\mathcal{A}}$  and  $\sigma_{\mathcal{A}}$  define Gleason measures by virtue of the results at the beginning of section 6. Note also that by Lemma 6.1,  $n_{\mathcal{A}}$  is absolutely continuous with respect to  $m$ . Furthermore, by virtue of Theorem 6.3 we can define  $n: P_S \rightarrow \mathbb{C}$  by:

$$n(T) = n_{\mathcal{A}}(T) = \int \mathcal{A}P_T dm$$

for any closed subspace  $T$  of  $H$ . Thus  $n$  is a measure, in particular a complex Gleason measure. To conclude, if we define an operator  $F$  on  $C[0, T; \mathcal{M}_b]$  by:

$$F(m) = \int \mathcal{A}P_T dm$$

and since  $m$  is a non-negative countably additive measure on the families of projections in  $P_S$ , by Definition 9.6, we obtain that  $F$  is a positive-operator valued measure. Thus we have constructed a complex Gleason measure that works as a positive-operator valued function. Hence the proof is finished.  $\square$

The results obtained help to construct a functional calculus approach for complex Gleason measures that will work as a positive-operator valued measure. In addition, the Nemytsky operator has the advantage of giving meaning to nonlinear terms of a class of nonlinear evolution equations under general conditions.

## 10. Applications to quantum mechanics and examples

Throughout the years, a rigorous development of the general formalism of quantum mechanics has been achieved by taking advantage of Hilbert space theory (see [1], [20], [21], [28], [38], [9], and [30]). In this section we intend to bring some of the concepts and results previously presented to the context of quantum mechanics and present some applications.

### 10.1. Applications of measure theory and Hilbert spaces to wave mechanics

The beginning of modern quantum mechanics, in which the famous physicist Schrödinger was a major exponent, was marked by the paper in which he proposed the formalism of wave mechanics, a concept that lies at the core of quantum mechanics [30]. In this paper we can observe that when discussing the physical interpretation of the so-called Schrödinger equation it is known that for each interval  $I$ :

$$P_t(I) = \int_I |\psi(x, t)|^2 dx$$

is the probability of finding a system in the state  $\psi(x, t)$  within  $I$  at a given time  $t$ .

Let  $(\Omega, \mathcal{B}, \mu)$  be a measure space, with the previous definition of probability it is natural to investigate *square integrable functions*, i.e. an extended complex-valued function  $f(x)$ ,  $x \in \Omega$ , defined almost everywhere on  $\Omega$  such that  $f(x)$  is measurable and  $|f(x)|^2$  is integrable on  $\Omega$ :

$$\int_{\Omega} |f(x)|^2 d\mu(x)$$

exists and is finite ([30]). We consider the space  $L^2(\Omega, \mu)$  of all complex-valued functions which are square integrable on  $\Omega$  with the equivalence relation given by almost everywhere equality; moreover denote the family of all equivalence classes with the usual symbol  $L^2(\Omega, \mu)$ . Furthermore  $L^2(\Omega, \mu)$  becomes a vector space and a Hilbert space with the inner product of  $f, g \in L^2(\Omega, \mu)$  defined by:

$$\langle f | g \rangle = \int_{\Omega} f^*(x)g(x)d\mu(x),$$

where  $f^*(x)$  denotes the complex conjugate.

With these concepts we shall introduce the structure used in quantum mechanics, more specifically in wave mechanics, to describe a system of  $n$  particles in which each particle is distinct. To start we assume that the  $n$  particles in our system move in three dimensions, and we will denote  $\mathbf{r}_k$  the position vector of the  $k^{th}$  particle. We can expand  $\mathbf{r}_k$  in terms of three orthonormal vectors  $\mathbf{p}_x$ ,  $\mathbf{p}_y$ ,  $\mathbf{p}_z$ , in a reference system of coordinates in the real Euclidean space  $\mathbb{R}^3$  as follows:

$$\mathbf{r}_k = x_k \mathbf{p}_x + y_k \mathbf{p}_y + z_k \mathbf{p}_z, \quad k = 1, 2, 3, \dots, n.$$

In wave mechanics it is postulated that the state of a system of  $n$  particles is given by the function  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  at any given time  $t$ , defined on the configuration space  $\mathbb{R}^{3n}$  of coordinate vectors  $\mathbf{r}_1, \dots, \mathbf{r}_n$ . Furthermore we shall assume that  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  is once continuously differentiable in  $t$ , and square integrable with respect to the measure space  $(\mathbb{R}^{3n}, \mathcal{B}^{3n}, \mu_l^{3n})$  (where  $\mathcal{B}^{3n}$  denotes the family of Borel sets in  $\mathbb{R}^{3n}$ ), i.e.  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t) \in L^2(\mathbb{R}^{3n})$ , and normalized to verify:

$$\int_{\mathbb{R}^{3n}} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)|^2 d\mathbf{r}_1 \dots d\mathbf{r}_n = 1.$$

In general, if a given function  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  represents a state of the system in question, then  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  is called a *wave function* of the corresponding system.

Note that the wave function  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  by itself does not have any physical meaning, rather we have that the measure given by:

$$P_t(B) = \int_B |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)|^2 d\mathbf{r}_1 \dots d\mathbf{r}_n, \quad B \in \mathcal{B}^{3n}$$

is interpreted as a probability measure;  $P_t(B)$  represents for every Borel set  $B$ , the probability of having the outcome of a measurement at time  $t$  of the positions  $\mathbf{r}_1, \dots, \mathbf{r}_n$  of the  $n$  particles of a system's state given by  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  within  $B$ .

In the context defined above it turns out that if a state is described by the wave function  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  at some given time  $t$ , then the function defined by  $c\psi_1(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$ , with the assumption that  $|c| = 1$  and  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t) = c\psi_1(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  almost everywhere with respect to the Lebesgue measure on  $\mathbb{R}^{3n}$ , describes the same state. Furthermore, this implies that each function in the equivalence class  $\psi(t) \in L^2(\mathbb{R}^{3n})$ , in which  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$  is contained, would be described by the same wave function  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t)$ . Finally, this previous result allows us to systematically formulate one of the basic assumptions of wave mechanics in a convenient way:

**POSTULATE 1:** The state of a system of  $n$  different particles is described at any time  $t$  by a normalized vector  $\psi(t)$  from the Hilbert space  $L^2(\mathbb{R}^{3n})$ . The time-dependent vector function  $c\psi(t)$ ,  $|c| = 1$ , represents the same state as  $\psi(t)$  (see Chapter II, section 5, p. 120 in [30]).

The reader may refer to [30] for more details, applications, interpretations of Hilbert spaces and measure theory concepts and basic assumptions of wave mechanics.

POSTULATE 2: Given the equation:

$$(10.1) \quad \left[ - \sum_{j=1}^{n-1} \frac{\hbar^2}{2M_j} \Delta'_j + V(\mathbf{r}'_1, \dots, \mathbf{r}'_{n-1}) \right] \psi((\mathbf{r}'_1, \dots, \mathbf{r}'_{n-1})) \\ = E_b \psi(\mathbf{r}'_1, \dots, \mathbf{r}'_{n-1}),$$

where  $M_j$  represents the mass of the  $j^{th}$  particle,  $V(\mathbf{r}'_1, \dots, \mathbf{r}'_{n-1})$  the potential of the particles interacting inside a system with no external forces,  $\Delta'_j = \frac{\partial^2}{\partial x_j'^2} + \frac{\partial^2}{\partial y_j'^2} + \frac{\partial^2}{\partial z_j'^2}$  and  $E_b$  the internal energy of the system.  $\mathbf{S}_p$  is the set of all eigenvalues of (10.1) that are the only internal energy values for which the  $n$  particle system of a bound state can assume. The closed linear subspace  $\mathcal{H}_b^{(n)}$  of  $L^2(\mathbb{R}^{3n})$ , spanned by all  $\psi \in L^2(\mathbb{R}^{3n})$  for which

$$\psi_{\mathbf{R}}(\mathbf{r}'_1, \dots, \mathbf{r}'_{n-1}) = \psi(\mathbf{r}_1, \dots, \mathbf{r}_n), \quad \psi_{\mathbf{R}} \in L^2(\mathbb{R}^{3(n-1)}),$$

is an eigenfunction of (10.1) for every  $\mathbf{R} \in \mathbb{R}^3$ , contains all the Hilbert vectors which can represent, at a given time  $t$ , a bound-state of the  $n$ -particle system interacting via the potential

$$V(\mathbf{r}_1, \dots, \mathbf{r}_{n-1}) = V(\mathbf{r}'_1, \dots, \mathbf{r}'_{n-1}).$$

An eigenvalue  $E_b$  of (10.1), with eigenfunction  $\psi \in L^2(\mathbb{R}^{3(n-1)})$ , is an internal bounded energy if  $E_b \in \mathbb{R}$  (for details see pages 120-128 in [30]).

Next we present some results in quantum mechanics to justify the two postulates discussed above.

**THEOREM 10.1** ([30, Theorem 5.2, Chapter 5]). *The inertial energy operator defined as*

$$(10.2) \quad H_i = - \sum_{j=1}^{n-1} \frac{\hbar^2}{2M_j} \Delta'_j + V(\mathbf{r}'_1, \dots, \mathbf{r}'_{n-1})$$

*defines Hermitian operators when applied to twice continuously differentiable functions, which together with their first derivatives vanish faster than  $|\mathbf{r}_k|^{-1}$ ,  $k = 1, \dots, n$ , as  $|\mathbf{r}_k| \rightarrow \infty$ .*

**PROOF.** See page 127 in [30]. □

Furthermore note that if  $\psi_1, \psi_2 \in L^2(\mathbb{R}^{3(n-1)})$  are eigenfunctions of (10.2) corresponding to the eigenvalues  $E_b^{(1)}$  and  $E_b^{(2)}$ , respectively, then we also have, by (10.2), that  $H_i\psi_1, H_i\psi_2 \in L^2(\mathbb{R}^{3(n-1)})$ . Thus as a consequence of Theorem 10.1,

$$E_b^{(1)*} \langle \psi_1 | \psi_2 \rangle = \langle H_i \psi_1 | \psi_2 \rangle = \langle \psi_1 | H_i \psi_2 \rangle = E_b^{(2)} \langle \psi_1 | \psi_2 \rangle.$$

If we have  $\psi_1 = \psi_2$ , and therefore  $E_b^{(1)} = E_b^{(2)} = E_b$  following the relation above we obtain  $E_b = E_b^*$ . Moreover given  $E_b^{(1)} \neq E_b^{(2)}$  we conclude  $\langle \psi_1 | \psi_2 \rangle = 0$ .

Lastly we state two theorems from quantum mechanics before proceeding to present some concrete examples:

**THEOREM 10.2** ([30, Theorem 5.3, Chapter 5]). *Each eigenvalue  $E$  of the internal energy operator  $H_i$  (as defined above), belonging to an eigenfunction  $\psi \in L^2(\mathbb{R}^{3(n-1)})$ , is a real number. Eigenfunctions corresponding to different eigenvalues are mutually orthogonal.*

It is clear, then, that for each eigenvalue there is at least one nonzero eigenvector, and that eigenvectors corresponding to different eigenvalues are orthogonal, that is, linearly independent. Also since  $L^2(\mathbb{R}^{3(n-1)})$  is separable, any orthogonal system of vectors contains at most a countable number of elements. As a conclusion we state our last theorem:

**THEOREM 10.3** ([30, Theorem 5.4, Chapter 5]). *The number of bound-state energy eigenvalues is at most countably infinite, i.e., the point energy spectrum  $\mathbf{S}_p$  contains a countable number of elements.*

## 10.2. Example 1. Relationship between the Gleason measure and the density operator

This example is presented in order to exemplify the relationship between Gleason measures and the density operator in quantum mechanics.

Gleason measures are very important results in modern mathematical foundations of quantum mechanics due to its strong implications on how probabilities can be introduced into quantum mechanics by taking the trace of the product between the projection operator and the density operator.

Suppose we have a system consisting of  $N$  orthonormal basis  $\{v_i, i = 1, 2, \dots\}$ . If the system is characterized by a single wave function,  $|\psi(t)\rangle$  at time  $t$ , then we have an expansion of the pure state  $|\psi(t)\rangle$  in the



orthonormal basis states:

$$(10.3) \quad |\psi(t)\rangle = \sum_i k_i(t) |v_i\rangle,$$

where the coefficients  $k_i(t)$  are given by  $\langle v_i | \psi(t) \rangle$ . Assuming that the state vectors  $|\psi(t)\rangle$  are normalized to unity then we have:

$$(10.4) \quad \langle \psi(t) | \psi(t) \rangle = \sum_i |k_i(t)|^2 = 1.$$

Let  $A$  be an observable (i.e. self-adjoint operator), then the matrix elements of  $A$  in the basis are given as:

$$(10.5) \quad A_{ij} = \langle v_j | A v_i \rangle = \langle A v_j | v_i \rangle = \langle v_j | A | v_i \rangle.$$

Equation (10.5) follows closely from the quantum mechanics notation of inner product. The expectation value of  $A$  at time  $t$  in the pure state  $|\psi(t)\rangle$  is:

$$\begin{aligned} \langle A \rangle &= \langle \psi(t) | A | \psi(t) \rangle \\ &= \sum_i \sum_j k_j^*(t) k_i(t) A_{ji} \\ &= \sum_i \sum_j \langle v_j | \psi(t) \rangle \langle \psi(t) | v_i \rangle A_{ji}, \end{aligned}$$

where  $*$  denotes the complex conjugate.  $\langle A \rangle$  is a quadratic expansion in the  $\{k_i\}$  coefficients that corresponds to the weighted average of all possible outcomes. The operator  $|\psi(t)\rangle \langle \psi(t)|$  has matrix elements which appears in the calculation of  $\langle A \rangle$ .

The density operator is formally defined as:

$$(10.6) \quad \rho(t) = |\psi(t)\rangle \langle \psi(t)|,$$

where  $|\psi(t)\rangle$  denotes a wave function. Please refer to section 2.2 of [10] for properties of the density operator. We recall from section 2 of this paper that the operator, (10.6) is a Hermitian operator which has the matrix elements

$$\rho_{ji}(t) = \langle v_j | \rho(t) | v_i \rangle = k_j(t) k_i^*(t),$$

where  $*$  denotes the complex conjugate. Since  $\psi(t)$  is normalized, we deduce from (10.4), that

$$1 = \sum_i |k_i(t)|^2 = \sum_i \rho_{ii}(t) = \text{Tr}(\rho(t)),$$

where  $\text{Tr}$  is the trace. Thus, the mean value of the observable  $A$  expressed using the density operator is given as:

$$\begin{aligned} \langle A \rangle(t) &= \sum_i \sum_j k_j^*(t) k_i^*(t) A_{ji} \\ &= \sum_i (\rho(t) A)_{ii} \\ (10.7) \qquad &= \text{Tr}(\rho(t) A). \end{aligned}$$

In general, the mean value of an arbitrary function  $F(A)$  is similarly obtained by replacing  $A$  in (10.7) by  $F(A)$ . Once  $\rho(t)$  is known, one can derive the statistical distribution of the results of the measurement of  $A$ .

In particular, if one specifies a state  $|\chi\rangle$ , then the probability of finding the system in the quantum state  $|\chi\rangle$  is  $\langle \chi | \rho | \chi \rangle$ .

The density operator is a practical tool when dealing with mixed states (i.e. statistical mixtures in which we have imperfect information about the system, for which we must perform statistical averages in order to describe the quantum observables). The density operator enables one to calculate the expectation of  $A$  and also provide information necessary to compute the probability of any outcome in any future measurement.

A practical application of the density operator formalism is to analyze the elastic scattering of the spin of particles ([10]). This analysis can be applied to the scattering of nucleons by pions. The density operator formalism is of much essence since it provides a statistical measure when dealing with mixed states as a result of the lack of information available on the system.

### 10.3. Example 2. The spin-1 particle

We present this example to emphasize the application of the density operator to the spin of a particle. Suppose the states of a system  $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$

form an orthonormal basis. If an observable  $A$  has the following properties:

$$\begin{aligned} A|v_1\rangle &= 1|v_1\rangle, \\ A|v_2\rangle &= 3|v_2\rangle, \\ A|v_3\rangle &= -2|v_3\rangle, \end{aligned}$$

then an expression for  $A$  in terms of its projection operators is given as:

$$(10.8) \quad A = 1|v_1\rangle\langle v_1| + 3|v_2\rangle\langle v_2| - 6|v_3\rangle\langle v_3|.$$

Using (10.5), the matrix representing the observable (10.8) is given as:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Let the spin-1 particle be in the states of amount  $-2/\sqrt{38}$ ,  $5/\sqrt{38}$  and  $3/\sqrt{38}$  corresponding to the eigenvectors  $v_1$ ,  $v_2$  and  $v_3$  respectively. Then the wave function described by (10.3) is given as:

$$(10.9) \quad |\psi(t)\rangle = \frac{-2}{\sqrt{38}}|v_1\rangle + \frac{5}{\sqrt{38}}|v_2\rangle + \frac{3}{\sqrt{38}}|v_3\rangle.$$

Given the state (10.9), the sum  $\langle\psi(t)|\rho(t)|\psi(t)\rangle$  enables one to compute the probability of the spin-1 particle in the state (10.9), where

$$\rho(t) = \begin{pmatrix} \frac{2}{19} & \frac{-5}{19} & \frac{-3}{19} \\ \frac{-5}{19} & \frac{25}{38} & \frac{15}{38} \\ \frac{-3}{19} & \frac{15}{38} & \frac{9}{38} \end{pmatrix}.$$

$\rho(t)$  is known as the density operator and was computed using (10.6).

The expectation value of the observable  $A$ , (10.9), expressed using the density operator (10.7) is given as:

$$\langle A \rangle(t) = \frac{61}{38}.$$

Thus the average value of  $A$  over the ensemble is  $61/38$ . This value represents the average of a collection of results obtained when measuring some observable  $A$ .

### 10.4. Example 3. A complex Gleason measure in quantum mechanics

Now consider a self-adjoint bounded linear operator  $A$  (i.e. observable) with eigenvalues  $\{\lambda_i | \lambda_{i+1}\} \lambda_i$  and corresponding eigenspaces  $\{S_i\}$ . For eigenspaces  $\{S_i\}$ , let  $\tilde{v}_i \in S_i, \tilde{v}_j \in S_j$ , then we conclude that  $\{S_i\}$  are orthogonal by Theorem 10.2.

The quantum mechanics notation is used in this section:  $\langle v_i | v_j \rangle$  denotes the inner product and  $\langle v_i | A | v_j \rangle = \langle v_i | A v_j \rangle = \langle A v_i | v_j \rangle$  ( $A$  is hermitian) as defined in (10.5). It is clear that when the operator  $A$  is written in the middle of the inner product it is automatically assumed to be self-adjoint. For more details on the quantum mechanics notation see for example [9].

Moreover we can restrict ourselves to the Hilbert space  $H$  given by the direct sum of  $\{S_i\}$ , we justify this using Theorem 10.3. Let us assume a particle in a superposition of pure states  $v_i \in S_i, \|v_i\| = 1$ , is defined by (10.3), where  $\sum_i |k_i|^2 = 1$  and  $|k_i|^2 = k_i^* k_i$  is called the probability amplitude of the state  $v_i$ . In other words the probability of the state  $v_i$ . We can define a complex Gleason measure in the following way:

$$\mu_{|\psi(t)\rangle}(P_i) := P_i |\psi(t)\rangle,$$

where  $P_i$  is a projector on a subspace  $S_i$ .  $\mu_{|\psi(t)\rangle}$  clearly defines a complex Gleason measure since it is additive on any given family of orthogonal projectors  $\{P_i\}$ ,  $P_i P_j = 0$  for  $i \neq j$ :

$$\mu_{|\psi(t)\rangle} \left( \sum_i P_i \right) = \sum_i \mu_{|\psi(t)\rangle}(P_i)$$

and has values in the complex plane. We can use the complex Gleason measure to recover  $k_i$  in the following way:

$$\int P_i d\mu_{|\psi(t)\rangle} = \mu_{|\psi(t)\rangle}(P_i) = P_i |\psi(t)\rangle = k_i.$$

### 10.5. Example 4. Connection of complex and real Gleason measures

Following example 3 in subsection 10.4, we will show the connection between the complex Gleason measure giving us  $k_i$ , and the real Gleason measure

giving  $|k_i|^2$  through the Nemytsky operator. Let  $A, v_i$  be as in subsection 10.4 and consider a finite superposition of pure states  $v_i$  as in (10.3). Denote  $S_i$  as:

$$S_i = \text{span}(v_i)$$

and  $P_i$  the projector on the linear subspace  $S_i$ . Using the quantum mechanics notation, the projector  $P_i$  will be described as:

$$P_i = |v_i\rangle\langle v_i|,$$

$$P_i|\psi(t)\rangle = \langle v_i|\psi(t)\rangle.$$

To be consistent with notation introduced in section 7, we will take

$$X = \mathbb{C},$$

$$Y = \mathbb{R}$$

and the measurable space will be the set  $S$  of all orthogonal subspaces  $S_i$  with sigma algebra generated by  $S_i$ . Consider the function  $f: S \rightarrow \mathbb{C}$  defined as:

$$f(S_i) = P_i|\psi(t)\rangle.$$

We can then write  $\mu_{|\psi(t)\rangle}$  from section 10.4 as:

$$\mu_{|\psi(t)\rangle}(S_i) = (\Lambda f)(S_i) = f(S_i) \cdot \Delta(S_i) = P_i|\psi(t)\rangle \cdot \dim(S_i) = P_i|\psi(t)\rangle,$$

where  $\mu_{|\psi(t)\rangle} \in \mathcal{M}_X$ , and  $\Delta$  is the measure given in section 6, that gives the dimension of  $S_i$ . We will define the following N-function  $g$  as:

$$g(S_i, y) = (P_i|\psi(t)\rangle)^* \cdot y,$$

then  $g$  satisfies the N-function properties given in definition 7.1. For fixed  $y \in \mathbb{C}$  the function

$$S \mapsto g(S, y) = \left( \sum_i (P_i|\psi(t)\rangle)^* \right) y, \quad S = \bigcup_i S_i,$$

is measurable. And for each  $S = \bigcup_i S_i$  the function

$$y \mapsto g(S, y) = \left( \sum_i (P_i|\psi(t)\rangle)^* \right) y, \quad S = \bigcup_i S_i,$$

is continuous.

The Nemytsky operator  $N_g$  is acting on  $f$  in the following way:

$$(N_g f)(S_i) = g(S_i, f(S_i)) = g(S_i, P_i|\psi(t)\rangle) = (P_i|\psi(t)\rangle)^*(P_i|\psi(t)\rangle)$$

Now we can define a real Gleason measure on  $S$  as:

$$\begin{aligned}\tilde{\mu}_{|\psi(t)\rangle}(S_i) &= (\Lambda N_g f)(S_i) \\ &= (P_i|\psi(t)\rangle)^*(P_i|\psi(t)\rangle) \cdot \Delta(S_i) \\ &= (P_i|\psi(t)\rangle)^*(P_i|\psi(t)\rangle) \cdot \dim(S_i) \\ &= (P_i|\psi(t)\rangle)^*(P_i|\psi(t)\rangle)\end{aligned}$$

and we obtain:

$$\begin{aligned}\mu_{|\psi(t)\rangle}(S_i) &= (\Lambda f)(S_i) = P_i|\psi(t)\rangle = k_i, \\ \tilde{\mu}_{|\psi(t)\rangle}(S_i) &= (\Lambda N_g f)(S_i) = (P_i|\psi(t)\rangle)^*(P_i|\psi(t)\rangle) = |k_i|^2,\end{aligned}$$

where  $|k_i|^2$  denotes the probability of the particle with wave function given by  $|\psi(t)\rangle$  being at the state  $v_i$ .

### 10.6. Example 5. The electron spin

In order to recall the importance of complex and real Gleason measures, we present an example involving an electron spin. Consider the observable  $A$  associated with an electron spin that is given by the spin quantum number  $m_l$  (for details see [28]). Now, the observable  $A$  has a set of discrete eigenvalues specifically, given by two values namely  $\lambda_\uparrow$  and  $\lambda_\downarrow$ , with corresponding eigenvectors given by  $v_1, v_2$  respectively. These represents the two possible states and values of an electron's spin. Moreover, we let  $v_1$  to represent an eigenvector in which the particle has a spin directed upwards and  $v_2$  an eigenvector with spin directed downwards. Now, given that a particle can be in a state of spin up or down, consider the particle in a superposition of states with a wave function described by (10.3) as:

$$|\psi(t)\rangle = k_1 v_1 + k_2 v_2,$$

where  $|k_1|^2 + |k_2|^2 = 1$  follows from (10.4) and  $|k_1|^2 = k_1^* k_1$  (respectively  $|k_2|^2$ ) is called the probability amplitude of the state  $v_1$  (respectively  $v_2$ ). In

particular, let a particle be in a state of amount  $^{12i}/_{13}$  in spin up and an amount of  $^5/_{13}$  in spin down; its wave function is then given by:

$$|\psi(t)\rangle = \frac{12}{13}iv_1 + \frac{5}{13}v_2.$$

Now, if we let  $S_\uparrow$  (respectively  $S_\downarrow$ ) be the eigenspace corresponding to the eigenvalue  $\lambda_\uparrow$  (respectively  $\lambda_\downarrow$ ), and define  $\mu_{|\psi(t)\rangle}, \tilde{\mu}_{|\psi(t)\rangle}$  as in example 4 (section 10.5), we then get

$$\mu_{|\psi(t)\rangle}(S_\uparrow) = (\Lambda f)(S_\uparrow) = P_\uparrow|\psi(t)\rangle = k_1 = \frac{12}{13}i.$$

Thus,

$$\begin{aligned} \tilde{\mu}_{|\psi(t)\rangle}(S_\uparrow) &= \tilde{\mu}_{|\psi(t)\rangle}(\Lambda N_g f)(S_\uparrow) \\ &= (P_\uparrow|\psi(t)\rangle)^*(P_\uparrow|\psi(t)\rangle) \\ &= |k_1|^2 \\ &= k_1^*k_1 \\ &= \left(-\frac{12}{13}i\right) \cdot \left(\frac{12}{13}i\right) \\ &= \frac{144}{169} \approx 0.85 \end{aligned}$$

and similarly,

$$\mu_{|\psi(t)\rangle}(S_\downarrow) = (\Lambda f)(S_\downarrow) = P_\downarrow|\psi(t)\rangle = k_2 = \frac{5}{13}.$$

Thus,

$$\tilde{\mu}_{|\psi(t)\rangle}(S_\downarrow) = \frac{25}{169} \approx 0.15,$$

where  $P_\uparrow$  (respectively  $P_\downarrow$ ) is a projector on the subspace  $S_\uparrow$  (respectively  $S_\downarrow$ ). Therefore, the probability that the particle described by the wave function  $|\psi(t)\rangle$  is at the pure state  $v_1$  (respectively  $v_2$ ) is 0.85 (respectively 0.15). To conclude this example we highlight that the possible values of the spin depend solely on the type of particle we are dealing with, in this case for simplicity we presented an electron to easily see the relation with Gleason measures.

### 10.7. Example 6. The positive-operator valued measure

To further exemplify the role of a positive-operator valued measure in quantum mechanics, we present the following simple example. Let  $H$  be a Hilbert space, recall that a POVM is a mapping that takes an element of the  $\sigma$ -algebra of subsets of  $H$  and maps it to a positive operator. Furthermore to every operator of this type, given a state there is a measure associated to the operator via the scalar product. For example let us consider the problem of measuring the orbital angular-momentum direction of an electron. The outcomes of such experiment are determined by the quantum number  $m_l$  with possible values  $-l, \dots, 0, \dots, +l$ , where  $l$  is the orbital quantum number. We can now define a positive-operator valued measure considering  $\Omega$  to be the Borel sigma algebra of the set  $\{-l, \dots, 0, \dots, +l\}$ , then the POVM is a map

$$\mathcal{E}: \Omega \rightarrow \mathcal{B}(H)$$

defined by:

$$\mathcal{E}(\{i\}) = P_i, \quad \mathcal{E}(\emptyset) = 0, \quad \mathcal{E}\left(\bigcup i\right) = \sum P_i, \quad -l \leq i \leq +l$$

for each integer number  $i$ , where  $P_i$  is the positive-operator associated to the orbital angular-momentum direction measurement. Let  $\rho$  be the density operator, associated with a given state and define:

$$\mu_\rho(\psi) = \text{Tr}(\mathcal{E}(\psi)\rho) \quad \forall \psi \in \Omega.$$

Then  $\mu_\rho$  defines a measure and in particular a probability measure. Moreover, if we take  $\{-l + 2\} \in \Omega$  then  $\mu_\rho(\{-l + 2\})$  would be the probability that when we measure  $\rho$  we will get the outcome  $-l + 2$ . Such relation clarifies the necessity of the empty set being mapped to 0, the whole space to 1 and lastly the operators being positive and add up to one.

We remark that the applications presented above are generalizations of some classical examples presented to incorporate complex measures and positive-operator valued measures (see examples 1, 2, 5 and 6). However, for examples 3 and 4, we used the complex Gleason measure to define the probability of a particle through the Nemytsky operator.

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