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INSERT A ROOT TO EXTRACT A ROOT OF QUINTIC QUICKLY

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Abstract. The usual way of solving a solvable quintic equation has been to establish more equations than unknowns, so that some relation among the coefficients comes up, leading to the solutions. In this paper, a relation among the coefficients of a principal quintic equation is established by effecting a change of variable and inserting a root to the quintic equation, and then equating odd-powers of the resulting sextic equation to zero. This leads to an even-powered sextic equation, or equivalently a cubic equation; thus one needs to solve the cubic equation.

We break from this tradition, rather factor the even-powered sextic equation in a novel fashion, such that the inserted root is identified quickly along with one root of the quintic equation in a quadratic factor of the form, $u^2 - g^2 = (u + g)(u - g)$. Thus there is no need to solve any cubic equation. As an extra benefit, this root is a function of only one coefficient of the given quintic equation.

1. Introduction

Consider a solvable quintic equation; what could be the simplest form of expression for one of its roots? May be the one, which involves only one coefficient. Is it possible to obtain such an expression? We shall make an attempt here.

First, let us gather some facts on quintics. After solving cubics and quartics, several mathematicians in seventeenth and eighteenth centuries struggled

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to solve general quintic equation in radicals, but with no success. In 1826, Abel (and later in 1832, Galois) showed that general polynomial equations of degree greater than four cannot be solved in radicals. With some condition imposed either on roots or coefficients, these equations become solvable and so are termed as solvable equations ([2]).

Notice that the general quintic equation,

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0,$$

contains all the power-terms of x, whereas a reduced (or depressed) quintic equation has no x^4 term, a principal quintic equation has no x^4 and x^3 terms, and a Bring-Jerrard quintic equation has no x^4, x^3, and x^2 terms ([1]). Using a simple linear transformation, we obtain a reduced quintic from the general one; using a quadratic Tschirnhaus transformation principal quintic is obtained; and Bring-Jerrard quintic is derived (quite laboriously!) using a quartic Tschirnhaus transformation ([1, 3]).

2. Root of a principal quintic equation

Coming back to our task of quickly obtaining a simplest possible form of expression for a root, let us consider a principal quintic equation,

$$x^5 + ax^2 + bx + c = 0, \tag{2.1}$$

where a, b, and c are non-zero coefficients. Letting $x = u + f$ in (2.1) yields,

$$(u + f)^5 + a(u + f)^2 + b(u + f) + c = 0, \tag{2.2}$$

where u is a new variable and f an unknown number. Expanding (2.2) and further rearranging it in descending powers of u yields,

$$u^5 + 5fu^4 + 10f^2u^3 + (10f^3 + a)u^2 + (5f^4 + 2af + b)u + f^5 + af^2 + bf + c = 0. \tag{2.3}$$

Inserting a root, $-g$, into the quintic equation (2.3) [which means multiplying (2.3) with $(u + g)$] and rearranging it in descending powers of u results in a sextic equation as shown below,

$$(2.4)\quad u^6 + (5f + g)u^5 + (10f^2 + 5fg)u^4 + (10f^3 + a + 10f^2 g)u^3 + [5f^4 + 2af + b + (10f^3 + a)g]u^2 + [f^5 + af^2 + bf + c + (5f^4 + 2af + b)g]u + (f^5 + af^2 + bf + c)g = 0.$$

Note that g is also an unknown number. Equating the coefficients of u^5, u^3, and u in (2.4) to zero not only makes (2.4) an even powered sextic equation as shown below,

$$(2.5)\qquad u^6 - 15f^2u^4 + (b - 3af - 45f^4)u^2 - 5f(f^5 + af^2 + bf + c) = 0,$$

but also results in the following three expressions in two unknowns (f and g):

$$(2.6)\qquad g = -5f,$$

$$(2.7)\qquad 10f^3 + a + 10f^2 g = 0,$$

$$(2.8)\qquad f^5 + af^2 + bf + c + (5f^4 + 2af + b)g = 0.$$

Eliminating g from (2.7) and (2.8) using (2.6) yields,

$$(2.9)\qquad f = (a/5)^{1/3}/2,$$

$$(2.10)\qquad c = 24f^5 + 9af^2 + 4bf.$$

We determine f from (2.9), and subsequently g from (2.6). Eliminating f from (2.10) using (2.9) yields,

$$(2.11)\qquad c = 12(a/5)^{5/3} + 2b(a/5)^{1/3}.$$

Expression (2.11) is the condition to be satisfied by the coefficients of quintic (2.1) to make it solvable. Notice that the sextic equation (2.5) is in effect a cubic equation in u^2, which can be solved obtaining all the six roots of u. Also note that one of the roots is the added root to the quintic equation (2.3), and the remaining five are the roots of (2.3). Subsequently, the roots of quintic equation (2.1) are obtained from the roots of (2.3) using the relation, $x = u + f$.

Instead of proceeding in this traditional manner, we are more interested in extracting one root of quintic equation (2.1) directly and quickly. For this purpose, we eliminate the coefficient a from the sextic equation (2.5) using the relation $a = 40f^3$ [see (2.9)], leading to:

$$u^6 - 15f^2u^4 + (b - 165f^4)u^2 - 5f(41f^5 + bf + c) = 0. \tag{2.12}$$

Use of $a = 40f^3$ in (2.10) yields,

$$c = 384f^5 + 4bf. \tag{2.13}$$

Now, using (2.13) we eliminate c from (2.12), which results in,

$$u^6 - 15f^2u^4 + (b - 165f^4)u^2 - 2125f^6 - 25bf^2 = 0. \tag{2.14}$$

If (2.14) is rearranged as below,

$$u^6 - 15f^2u^4 - 165f^4u^2 - 2125f^6 + b(u^2 - 25f^2) = 0,$$

we recognize that it can be factored as,

$$(u^2 - 25f^2)(u^4 + 10u^2f^2 + 85f^4 + b) = 0. \tag{2.15}$$

The quadratic factor in (2.15) can be factored again as, $(u-5f)(u+5f)$, and since $g = -5f$ [see (2.6)] further it results in $(u+g)(u-g)$; so (2.15) becomes

$$(u+g)(u-g)(u^4 + 10u^2f^2 + 85f^4 + b) = 0. \tag{2.16}$$

Notice that the first factor in (2.16) is due to the inserted root, $-g$ into the quintic equation (2.3), and therefore after discarding this factor we are left with the factored quintic equation (2.3) as shown below,

$$(u-g)(u^4 + 10u^2f^2 + 85f^4 + b) = 0. \tag{2.17}$$

Now, equating the first factor in (2.17) to zero yields a root (say u_1) of (2.3) as, $u_1 = g = -5f$, and since $x = u + f$, we obtain a root (x_1) of quintic equation (2.1) as, $x_1 = -4f$; and further use of (2.9) yields,

$$x_1 = -2(a/5)^{1/3}. \tag{2.18}$$

Thus we note that one root of principal quintic equation (2.1) can be extracted quickly from (2.18), provided the quintic satisfies the condition stipulated in (2.11). Also notice that (2.18) involves only one coefficient (a) of the given quintic equation (2.1).

3. Numerical example

Consider the principal quintic equation, $x^5 + 5x^2 + 6x + 24 = 0$, which satisfies the condition (2.11). From (2.18) we obtain a root as, $x_1 = -2$. The remaining four roots can be determined by equating the biquadratic factor in (2.17) to zero, and solving it using quadratic formula. To do this, first we determine f from (2.9) as: $f = 0.5$.

We obtain the biquadratic equation [see (2.17)] as, $u^4+2.5u^2+11.3125 = 0$, solving which, we determine the two roots: $u^2 = -1.25 \pm 3.122498999...i$. Subsequently the four roots are obtained as:

$$u_2,\ u_3 = \pm(1.027960605... + 1.518783396...i),$$

$$u_4,\ u_5 = \pm(1.027960605... - 1.518783396...i).$$

From the above roots, the roots of x are obtained through $x = u + f$,

$$x_2,\ x_3 = 0.5 \pm (1.027960605... + 1.518783396...i),$$

$$x_4,\ x_5 = 0.5 \pm (1.027960605... - 1.518783396...i).$$

4. Summary

This paper has presented a method for solving a solvable principal quintic equation, in which one root of the given quintic equation is extracted quickly. In this method, the given quintic equation is transformed to another quintic and a root is inserted to it resulting in a sextic equation. The odd-powers in the sextic equation are equated to zero, leading to an even-powered sextic equation (or equivalently, a cubic equation); this also yields a condition to be satisfied by the coefficients to make the quintic solvable. The even-powered sextic equation can be solved in a traditional manner by using Cardan's method.

Instead, this paper has employed a novel decomposition technique, in which the even-powered sextic equation is factored into a quartic factor and a quadratic factor, which is of the form, $u^2 - g^2 = (u + g)(u - g)$, revealing the inserted root and a root of the transformed quintic equation. One can notice that the root of the given quintic equation is obtained without solving the cubic equation. Incidentally, the expression for this root involves only one coefficient of the quintic.

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