

## D-HOMOTHETICALLY DEFORMED KENMOTSU METRIC AS A RICCI SOLITON

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**Abstract.** In this paper we study the nature of Ricci solitons in  $D$ -homothetically deformed Kenmotsu manifolds. We prove that  $\eta$ -Einstein Kenmotsu metric as a Ricci soliton remains  $\eta$ -Einstein under  $D$ -homothetic deformation and the scalar curvature remains constant.

### 1. Introduction

One of the important topics in the study of almost contact metric manifolds is the study of Ricci flow and Ricci solitons. A Ricci soliton is a Riemannian metric  $g$  on a manifold  $M$  together with a vector field  $V$  such that

$$(1.1) \quad (\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

where  $\mathcal{L}_V$ ,  $S$  and  $\lambda$  denote the Lie derivative along  $V$ , Ricci tensor and a constant. A Ricci soliton is said to be shrinking or steady or expanding if  $\lambda$  is negative, zero or positive, respectively. A Ricci soliton is said to be a gradient Ricci soliton if the vector field  $V$  is gradient of some smooth function  $f$  on  $M$ .

Sharma ([11]) initiated the study of Ricci solitons in contact Riemannian geometry. Ghosh and Sharma ([5], [6]), Sharma ([11]) established results by considering  $K$ -contact, Kenmotsu, Sasakian and  $(\kappa, \mu)$ -contact metrics as

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Ricci solitons. Bejan and Crasmareanu ([1]) extended the study of Ricci solitons to paracontact manifolds. De and others ([15], [8], [9]) studied Ricci solitons in  $f$ -Kenmotsu manifolds and Kenmotsu manifolds. In [10] authors analyze the behaviour of trans-Sasakian manifolds under  $D$ -homothetic deformations. Several authors, e.g. Nagaraja and Premalatha ([7]), De and Ghosh ([4]) studied the behaviour of  $K$ -contact, normal almost contact metric manifolds under  $D$ -homothetic deformations. We make use of the invariance of certain contact structures under  $D$ -homothetic deformations to study Ricci solitons.

This paper is structured as follows: after a brief review of Kenmotsu manifolds in section 2, we study  $D$ -homothetically deformed Kenmotsu metrics as Ricci solitons in section 3.

## 2. Preliminaries

A  $(2n+1)$ -dimensional smooth manifold  $M$  is said to be an almost contact metric manifold if it admits an almost contact metric structure  $(\phi, \xi, \eta, g)$  consisting of a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$ , a 1-form  $\eta$  and a Riemannian metric  $g$  compatible with  $(\phi, \xi, \eta)$  satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad g(X, \xi) = \eta(X), \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0,$$

and

$$(2.1) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

An almost contact metric manifold is said to be a Kenmotsu manifold ([2]) if

$$(2.2) \quad (\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

$$(2.3) \quad \begin{aligned} \nabla_X \xi &= X - \eta(X)\xi, \\ (\nabla_X \eta)Y &= g(\nabla_X \xi, Y), \end{aligned}$$

where  $\nabla$  denotes the Riemannian connection of  $g$ .

In a Kenmotsu manifold the following relations hold ([3]):

$$(2.4) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(2.5) \quad S(X, \xi) = -2n\eta(X),$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y),$$

for any vector fields  $X, Y, Z$  on  $M$ , where  $R$  denotes the curvature tensor of type  $(1, 3)$  on  $M$ .

A vector field  $V$  on a Kenmotsu manifold is said to be conformal Killing vector field ([14]) if

$$(2.6) \quad (\mathcal{L}_V g)(X, Y) = 2\rho g(X, Y),$$

where  $\rho$  is a function on the manifold.

Let  $(g, V, \lambda)$  be a Ricci soliton in a 3 dimensional Kenmotsu manifold  $M$ . Then from (2.6) and (1.1), we have

$$(2.7) \quad S(X, Y) = -(\lambda + \rho)g(X, Y),$$

which yields

$$(2.8) \quad QX = -(\lambda + \rho)X,$$

$$S(X, \xi) = -(\lambda + \rho)\eta(X),$$

$$(2.9) \quad r = -3(\lambda + \rho),$$

where  $Q$  is the Ricci operator and  $r$  is the scalar curvature on  $M$ .

### 3. Ricci solitons in Kenmotsu manifolds under $D$ -homothetic deformations

Let  $(M, \phi, \xi, \eta, g)$  be an almost contact metric manifold, where  $g$  is a Ricci soliton. The  $D$ -homothetic deformation ([13]) on  $M$  is given by

$$(3.1) \quad \phi^* = \phi, \quad \xi^* = \frac{1}{a}\xi, \quad \eta^* = a\eta, \quad g^* = ag + a(a-1)\eta \otimes \eta$$

for a positive constant  $a$ . If  $(M, \phi, \xi, \eta, g)$  is an almost contact metric structure with contact form  $\eta$ , then  $(M, \phi^*, \xi^*, \eta^*, g^*)$  is also an almost contact metric

structure ([13]). Now we recall the Ricci tensor of a Kenmotsu manifold transforms under a  $D$ -homothetic deformation ([10]) as

$$(3.2) \quad S^*(X, Y) = S(X, Y) + \frac{2n(a-1)}{a} \{g(X, Y) + (a - a^2 - 1)\eta(X)\eta(Y)\}.$$

Taking the Lie derivative of  $g^* = ag + a(a-1)\eta \otimes \eta$  along  $V$  and using (3.1) and (3.2), we obtain

$$\begin{aligned} (3.3) \quad (\mathcal{L}_V g^*)(X, Y) + 2S^*(X, Y) + 2\lambda g^*(X, Y) \\ = a(\mathcal{L}_V g)(X, Y) + a(a-1)\{(\mathcal{L}_V \eta)(X)\eta(Y) + \eta(X)\mathcal{L}_V \eta(Y)\} \\ + 2S(X, Y) + \frac{4n(a-1)}{a} \{g(X, Y) + (a - a^2 - 1)\eta(X)\eta(Y)\} \\ + 2\lambda a \{g(X, Y) + (a-1)\eta(X)\eta(Y)\}. \end{aligned}$$

We Lie-differentiate  $\eta(\xi) = 1$  along  $V$  to get

$$(3.4) \quad (\mathcal{L}_V \eta)(\xi) + \eta(\mathcal{L}_V \xi) = 0.$$

Also Lie-differentiation of  $g(\xi, \xi) = 1$  along  $V$  gives

$$(3.5) \quad (\mathcal{L}_V g)(\xi, \xi) + 2\eta(\mathcal{L}_V \xi) = 0.$$

Further, setting  $X = Y = \xi$  in (1.1) and using (2.5), we obtain

$$(3.6) \quad (\mathcal{L}_V g)(\xi, \xi) = 4n - 2\lambda.$$

Using (3.6), equation (3.5) yields

$$(3.7) \quad \eta(\mathcal{L}_V \xi) = \lambda - 2n.$$

Now, (3.4) yields

$$(\mathcal{L}_V \eta)(\xi) = 2n - \lambda.$$

By putting  $Y = \xi$  in (1.1), we obtain

$$(3.8) \quad (\mathcal{L}_V \eta)(X) = g(X, \mathcal{L}_V \xi) - 2S(X, \xi) - 2\lambda\eta(X).$$

We know that  $\mathcal{L}_V \xi = \eta(\mathcal{L}_V \xi)\xi$  ([12]) and using (2.5), (3.7) in (3.8), we get

$$(3.9) \quad (\mathcal{L}_V \eta)(X) = (2n - \lambda)\eta(X).$$

By hypothesis  $(\mathcal{L}_V g)(X, Y) = -2(S(X, Y) + \lambda g(X, Y))$  and with the use of (3.9), (3.3) reduces to

$$\begin{aligned} (\mathcal{L}_V g^*)(X, Y) + 2S^*(X, Y) + 2\lambda g^*(X, Y) \\ = -2(a-1)[S(X, Y) - \frac{2n}{a}\{g(X, Y) + (a-1)\eta(X)\eta(Y)\}], \end{aligned}$$

i.e  $g^*$  is a Ricci soliton if and only if

$$(3.10) \quad S(X, Y) = \frac{2n}{a}\{g(X, Y) + (a-1)\eta(X)\eta(Y)\}.$$

Therefore, we have the following theorem.

**THEOREM 3.1.** *Under D-homothetic deformation, a Kenmotsu metric which is  $\eta$ -Einstein Ricci soliton remains  $\eta$ -Einstein Ricci soliton.*

Contracting (3.10), we have

$$(3.11) \quad r = \frac{2n}{a}\{2n + a\}.$$

Let us now use the formula ([11])

$$(3.12) \quad \mathcal{L}_V r = -\Delta r + 2R_{ij}R^{ij} + 2\lambda r.$$

As  $r$  is a constant, we get

$$(3.13) \quad R_{ij}R^{ij} = -\lambda r.$$

On contracting (3.2), we obtain

$$(3.14) \quad r^* = r + \frac{2n(a-1)}{a}\{2n + a - a^2\}.$$

By substituting (3.11) in (3.14), we have

$$(3.15) \quad r^* = 2n(2n + 2a - a^2).$$

Thus, we state the following:

**THEOREM 3.2.** *An  $\eta$ -Einstein Kenmotsu metric as a Ricci soliton remains  $\eta$ -Einstein Ricci soliton and in this case the scalar curvature of a D-homothetically deformed Kenmotsu manifold is constant.*

Using (3.11), (3.13) becomes

$$(3.16) \quad R_{ij}R^{ij} = -\frac{2n\lambda}{a}\{2n+a\}.$$

Analogously to the formula (3.12), we write

$$\mathcal{L}_V r^* = -\Delta r^* + 2R_{ij}^*(R^*)^{ij} + 2\lambda r^*.$$

From (3.15),  $r^*$  is a constant, so we get

$$(3.17) \quad R_{ij}^*(R^*)^{ij} = -\lambda r^*.$$

By making use of (3.14) and (3.11), (3.17) becomes

$$(3.18) \quad R_{ij}^*(R^*)^{ij} = -\lambda r - \frac{2n\lambda(a-1)}{a}\{2n+a-a^2\}.$$

Comparing the above with (3.2), we get

$$(3.19) \quad R_{ij}^*(R^*)^{ij} = R_{ij}R^{ij} + \frac{4n^2(a-1)^2}{a^2}[\{g_{ij} + (a-a^2-1)\eta_i\eta_j\}\{g^{ij} + (a-a^2-1)\eta^i\eta^j\}].$$

After simplification, equation (3.19) gives

$$(3.20) \quad R_{ij}^*(R^*)^{ij} = R_{ij}R^{ij} + \frac{4n^2(a-1)^2}{a^2}\{2n+a^2(a-1)^2\}.$$

In view of (3.18) and (3.20), using (3.16), we obtain

$$\lambda = \frac{2n(1-a)\{2n+a^2(a-1)^2\}}{a\{2n+a(1-a)\}}.$$

Thus, we can state the following:

**THEOREM 3.3.** *A Ricci soliton in a D-homothetically deformed Kenmotsu manifold is expanding for  $a < 1$ .*

Since in a three-dimensional Riemannian manifold the conformal curvature tensor  $C$  vanishes, we have

$$(3.21) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY \\ + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y],$$

where  $R$  is Riemannian curvature tensor of type (1,3).

Using (2.7), (2.8), (2.9) in (3.21) and by putting  $Z = \xi$ , we get

$$(3.22) \quad R(X, Y)\xi = \frac{(\lambda + \rho)}{2}\{\eta(X)Y - \eta(Y)X\}.$$

By comparing (2.4) and (3.22), we obtain

$$\lambda + \rho = 2.$$

Thus, we have

**THEOREM 3.4.** *If the generating vector field  $V$  is a conformal Killing vector field with associated function  $\rho$ , then the Ricci soliton in a three-dimensional Kenmotsu manifold is shrinking or expanding or steady if  $\rho > 2$  or  $\rho < 2$  or  $\rho = 2$ , respectively.*

**EXAMPLE 3.1.** We consider the three-dimensional manifold

$$M = \{(x, y, z) \in R^3; z \neq 0\},$$

where  $(x, y, z)$  are the standard coordinates in  $R^3$ . The vector fields

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^z \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of  $M$ . Let  $g$  be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Let  $\eta$  be the 1-form defined by  $\eta(Z) = g(Z, E_3)$  for any  $Z \in \chi(M)$ . Let  $\phi$  be the  $(1, 1)$  tensor field defined by  $\phi E_1 = E_2, \phi E_2 = -E_1, \phi E_3 = 0$ . Then using the linearity of  $\phi$  and  $g$  we have

$$\eta(E_3) = 1, \quad \phi^2(Z) = -Z + \eta(Z)E_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any  $Z, W \in \chi(M)$ . Thus, for  $E_3 = \xi$ ,  $(\phi, \xi, \eta, g)$  defines an almost contact metric structure on  $M$ .

Let  $\nabla$  be the Levi-Civita connection with respect to the metric  $g$ . Then we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$

The Riemannian connection  $\nabla$  of the metric  $g$  is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By Koszul's formula, we get

$$\begin{aligned} \nabla_{E_1} E_3 &= -E_1, & \nabla_{E_2} E_3 &= -E_2, & \nabla_{E_3} E_3 &= 0, \\ \nabla_{E_1} E_2 &= 0, & \nabla_{E_2} E_2 &= E_3, & \nabla_{E_3} E_2 &= 0, \\ \nabla_{E_1} E_1 &= E_3, & \nabla_{E_2} E_1 &= 0, & \nabla_{E_3} E_1 &= 0. \end{aligned}$$

From the above expressions it follows that the manifold satisfies (2.1), (2.2) and (2.3) for  $\xi = E_3$ . Hence, the manifold is a Kenmotsu manifold. With the help of the above results we can verify the following results:

$$\begin{aligned} R(E_1, E_1)E_1 &= 0, & R(E_1, E_2)E_2 &= -E_1, & R(E_1, E_3)E_3 &= -E_1, \\ R(E_2, E_1)E_1 &= -E_2, & R(E_2, E_2)E_2 &= 0, & R(E_2, E_3)E_3 &= -E_2, \\ R(E_3, E_1)E_1 &= -E_3, & R(E_3, E_2)E_2 &= -E_3, & R(E_3, E_3)E_3 &= 0. \end{aligned}$$

From the above expressions of the curvature tensor, we obtain the non-zero components of Ricci tensor  $S$  as follows:

$$S(E_1, E_1) = g(R(E_1, E_2)E_2, E_1) + g(R(E_1, E_3)E_3, E_1) = -2.$$

Similarly, we have

$$S(E_2, E_2) = S(E_3, E_3) = -2.$$



For  $V = e^{-z}E_3$ , we have

$$(3.23) \quad (\mathcal{L}_V g)(E_i, E_i) = -2e^{-z}.$$

Now, by taking  $X = Y = E_i$  in (1.1), where  $i = 1, 2, 3$ , and by virtue of the above equations, we have that  $g$  is a Ricci soliton for  $\lambda = e^{-z} + 2$ . Here  $\lambda$  is positive for all  $z$ . Hence, the soliton is expanding.

Equation (3.23) can be written as  $(\mathcal{L}_V g)(E_i, E_i) = 2\rho g(E_i, E_i)$ , where  $\rho = -e^{-z}$ , i.e.  $\lambda + \rho = 2$ .

In this example  $\rho < 2$  for all values of  $z$ . This verifies Theorem 3.4.

Suppose  $(g^*, V, \lambda)$  is a Ricci soliton, where  $g^*$  is obtained by  $D$ -homothetic change of a three-dimensional Kenmotsu metric  $g$ . Then

$$(\mathcal{L}_V g^*)(X, Y) + 2S^*(X, Y) + 2\lambda g^*(X, Y) = 0.$$

Now, by taking the Lie derivative of  $g^* = ag + a(a-1)\eta \otimes \eta$  along  $V$  and using (3.9), we obtain

$$(3.24) \quad a\{(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y)\} + 4a(a-1)\eta(X)\eta(Y) \\ + 2(1-a)S(X, Y) + \frac{4(a-1)}{a}\{g(X, Y) + (a-a^2-1)\eta(X)\eta(Y)\} = 0.$$

By using (1.1) and (2.7), (3.24) becomes

$$(3.25) \quad \{\lambda + \rho + \frac{2}{a}\}g(X, Y) + \{2 - \frac{2}{a}\}\eta(X)\eta(Y) = 0.$$

Putting  $X = Y = \xi$  in (3.25), we get

$$\lambda + \rho = -2.$$

**THEOREM 3.5.** *Under  $D$ -homothetic deformation, Ricci soliton in a three-dimensional Kenmotsu manifold with the generating vector field  $V$  as a conformal Killing vector field and  $\rho$  as associated function is expanding or shrinking or steady if  $\rho < -2$  or  $\rho > -2$  or  $\rho = -2$ , respectively.*

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