

## ON A NEW ONE PARAMETER GENERALIZATION OF PELL NUMBERS

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**Abstract.** In this paper we present a new one parameter generalization of the classical Pell numbers. We investigate the generalized Binet's formula, the generating function and some identities for  $r$ -Pell numbers. Moreover, we give a graph interpretation of these numbers.

### 1. Introduction

The Pell sequence  $\{P_n\}$  is one of the special cases of sequences  $\{a_n\}$  which are defined recurrently as a linear combination of the preceding  $k$  terms

$$(1.1) \quad a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k} \quad \text{for } n \geq k,$$

where  $k \geq 2$ ,  $b_i$  are integers,  $i = 1, 2, \dots, k$  and  $a_0, a_1, \dots, a_{k-1}$  are given numbers.

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By recurrence (1.1) for  $k = 2$  we get (among others) the well-known recurrences:

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, & F_0 &= 0, & F_1 &= 1 & \text{(Fibonacci numbers),} \\ L_n &= L_{n-1} + L_{n-2}, & L_0 &= 2, & L_1 &= 1 & \text{(Lucas numbers),} \\ J_n &= J_{n-1} + 2J_{n-2}, & J_0 &= 0, & J_1 &= 1 & \text{(Jacobsthal numbers),} \\ P_n &= 2P_{n-1} + P_{n-2}, & P_0 &= 0, & P_1 &= 1 & \text{(Pell numbers).} \end{aligned}$$

The first ten terms of the Pell sequence are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985. The  $n$ -th Pell number is explicitly given by the Binet-type formula

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}} \quad \text{for } n \geq 0.$$

Moreover, the Pell numbers are defined by the following formula

$$P_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 2^k.$$

The matrix generator of the sequence  $\{P_n\}$  is  $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ . It is known that

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Hence we get the well-known formula (Cassini's identity)  $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$ . Another interesting properties of the Pell numbers are given in [4].

In the literature there are some generalizations of the Pell numbers. We recall some of them. In [5] the authors introduced  $p$ -Pell numbers  $P_p(n)$  defined by the following relation:  $P_p(n) = 2P_p(n-1) + P_p(n-p-1)$  for  $p = 0, 1, 2, \dots$  and  $n \geq p+2$  with  $P_p(1) = a_1$ ,  $P_p(2) = a_2$ ,  $\dots$ ,  $P_p(p+1) = a_{p+1}$ , where  $a_1, a_2, \dots, a_{p+1}$  are integers, real or complex numbers. Another generalization of the Pell numbers is given in [1], [2]: the  $k$ -Pell numbers  $\{P_{k,n}\}$  are defined recurrently by  $P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$  for  $k \geq 1$  and  $n \geq 1$  with  $P_{k,0} = 0$ ,  $P_{k,1} = 1$ .

In [6] there was presented  $k$ -distance Pell sequence defined as follows:  $P_k(n) = 2P_k(n-1) + P_k(n-k)$  for  $n \geq k$  with  $P_k(0) = 0$ ,  $P_k(n) = 2^{n-1}$  for  $n = 1, 2, \dots, k-1$ . Another interesting generalizations of the Pell numbers can be found in [9].

In this paper we introduce a new one parameter generalization of Pell numbers.

## 2. The $r$ -Pell numbers and some basic properties

Let  $n \geq 0$ ,  $r \geq 1$  be integers. Define  $r$ -Pell sequence  $\{P(r, n)\}$  by the following recurrence relation

$$(2.1) \quad P(r, n) = 2^r P(r, n-1) + 2^{r-1} P(r, n-2) \quad \text{for } n \geq 2$$

with initial conditions  $P(r, 0) = 2$ ,  $P(r, 1) = 1 + 2^{r+1}$ .

It is easily seen that  $P(1, n) = P_{n+2}$ . By (2.1) we obtain

$$\begin{aligned} P(r, 0) &= 2, \\ P(r, 1) &= 1 + 2^{r+1}, \\ P(r, 2) &= 2^{r+1} + 2 \cdot 4^r, \\ P(r, 3) &= 2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r, \\ P(r, 4) &= \frac{3}{2} \cdot 4^r + 4 \cdot 8^r + 2 \cdot 16^r. \end{aligned}$$

Now we present the Binet's formula, which allows us to express the  $r$ -Pell numbers in function of the roots  $r_1$  and  $r_2$  of the following characteristic equation, associated with the recurrence relation (2.1)

$$(2.2) \quad x^2 - 2^r x - 2^{r-1} = 0.$$

Then

$$(2.3) \quad r_1 = \frac{2^r + \sqrt{4^r + 2^{r+1}}}{2}, \quad r_2 = \frac{2^r - \sqrt{4^r + 2^{r+1}}}{2}.$$

PROPOSITION 2.1 (Binet's formula). *Let  $n \geq 0$ ,  $r \geq 1$  be integers. Then*

$$(2.4) \quad P(r, n) = C_1 r_1^n + C_2 r_2^n,$$

where  $r_1, r_2$  are given by (2.3) and

$$C_1 = 1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}, \quad C_2 = 1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}.$$

PROOF. The general term of the sequence  $\{P(r, n)\}$  may be expressed in the following form

$$P(r, n) = C_1 r_1^n + C_2 r_2^n$$

for some coefficients  $C_1$  and  $C_2$ . Using initial conditions of the recurrence (2.1), we obtain the following system of two linear equations

$$\begin{cases} C_1 + C_2 = 2, \\ C_1 r_1 + C_2 r_2 = 1 + 2^{r+1}. \end{cases}$$

Hence

$$C_1 = 1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}} \quad \text{and} \quad C_2 = 1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}},$$

which ends the proof.  $\square$

Since  $r_1$  and  $r_2$  are the roots of equation (2.2), we have

$$(2.5) \quad r_1 + r_2 = 2^r,$$

$$(2.6) \quad r_1 - r_2 = \sqrt{4^r + 2^{r+1}},$$

$$(2.7) \quad r_1 r_2 = -2^{r-1}.$$

Moreover, by simple calculations, we get

$$(2.8) \quad C_1 C_2 = -\frac{1}{4^r + 2^{r+1}},$$

$$(2.9) \quad C_1 r_2 + C_2 r_1 = -1.$$

### 3. Some identities for the sequence $\{P(r, n)\}$

In this section we present some properties and identities for the  $r$ -Pell numbers. They generalize known results for classical Pell numbers.

THEOREM 3.1. *Let  $r$  be a positive integer. Then*

$$\lim_{n \rightarrow \infty} \frac{P(r, n+1)}{P(r, n)} = \frac{2^r + \sqrt{4^r + 2^{r+1}}}{2}.$$

PROOF. Using Proposition 2.1, we have

$$\lim_{n \rightarrow \infty} \frac{P(r, n+1)}{P(r, n)} = \lim_{n \rightarrow \infty} \frac{C_1 r_1^{n+1} + C_2 r_2^{n+1}}{C_1 r_1^n + C_2 r_2^n} = \lim_{n \rightarrow \infty} \frac{C_1 r_1 + C_2 r_2 \left(\frac{r_2}{r_1}\right)^n}{C_1 + C_2 \left(\frac{r_2}{r_1}\right)^n}.$$

Since  $\lim_{n \rightarrow \infty} \left(\frac{r_2}{r_1}\right)^n = 0$ , we get

$$\lim_{n \rightarrow \infty} \frac{P(r, n+1)}{P(r, n)} = r_1 = \frac{2^r + \sqrt{4^r + 2^{r+1}}}{2}. \quad \square$$

THEOREM 3.2 (Cassini's identity). *Let  $n, r$  be positive integers. Then*

$$(3.1) \quad P(r, n+1)P(r, n-1) - P^2(r, n) = (-1)^n 2^{(r-1)(n-1)}.$$

PROOF. By Binet's formula (2.4) we obtain

$$\begin{aligned} P(r, n+1)P(r, n-1) - P^2(r, n) &= (C_1 r_1^{n+1} + C_2 r_2^{n+1})(C_1 r_1^{n-1} + C_2 r_2^{n-1}) - (C_1 r_1^n + C_2 r_2^n)^2 \\ &= C_1 C_2 (r_1 r_2)^n \left(\frac{r_1}{r_2} + \frac{r_2}{r_1} - 2\right) = C_1 C_2 (r_1 r_2)^{n-1} (r_1 - r_2)^2, \end{aligned}$$

where  $r_1, r_2$  are given by (2.3).

Using formulas (2.8), (2.7) and (2.6), we have

$$P(r, n+1)P(r, n-1) - P^2(r, n) = -(-2^{r-1})^{n-1} = (-1)^n 2^{(r-1)(n-1)}. \quad \square$$

By formula (3.1), considering  $r = 1$  and taking into account that  $P(1, n) = P_{n+2}$ , we obtain Cassini's identity for the classical Pell numbers.

COROLLARY 3.3. *For  $n \geq 1$ ,  $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$ .*

The next theorem presents a summation formula for the  $r$ -Pell numbers.

THEOREM 3.4. *Let  $n, r$  be positive integers. Then*

$$\sum_{i=0}^{n-1} P(r, i) = \frac{P(r, n) + 2^{r-1}P(r, n-1) - 3}{3 \cdot 2^{r-1} - 1}.$$

PROOF. Using formula (2.4), we have

$$\begin{aligned} \sum_{i=0}^{n-1} P(r, i) &= \sum_{i=0}^{n-1} (C_1 r_1^i + C_2 r_2^i) = C_1 \frac{1 - r_1^n}{1 - r_1} + C_2 \frac{1 - r_2^n}{1 - r_2} \\ &= \frac{C_1 + C_2 - (C_1 r_2 + C_2 r_1) - (C_1 r_1^n + C_2 r_2^n) + r_1 r_2 (C_1 r_1^{n-1} + C_2 r_2^{n-1})}{1 - (r_1 + r_2) + r_1 r_2}. \end{aligned}$$

By Binet's formula we get

$$\sum_{i=0}^{n-1} P(r, i) = \frac{C_1 + C_2 - (C_1 r_2 + C_2 r_1) - P(r, n) + r_1 r_2 P(r, n-1)}{1 - (r_1 + r_2) + r_1 r_2}.$$

By (2.9), (2.7) and (2.5) we obtain

$$\sum_{i=0}^{n-1} P(r, i) = \frac{P(r, n) + 2^{r-1}P(r, n-1) - 3}{3 \cdot 2^{r-1} - 1}. \quad \square$$

Using twice the recurrence (2.1), we obtain the following result.

PROPOSITION 3.5. *Let  $n, r$  be integers such that  $n \geq 4$ ,  $r \geq 1$ . Then*

$$P(r, n) = (8^r + 4^r)P(r, n-3) + (2^{3r-1} + 2^{2r-2})P(r, n-4).$$

THEOREM 3.6. *The generating function of the sequence  $\{P(r, n)\}$  has the following form*

$$f(x) = \frac{2 + x}{1 - 2^r x - 2^{r-1} x^2}.$$

PROOF. Assuming that the generating function of the sequence  $\{P(r, n)\}$  has the form  $f(x) = \sum_{n=0}^{\infty} P(r, n)x^n$ , we get

$$\begin{aligned}
 (1 - 2^r x - 2^{r-1} x^2)f(x) &= (1 - 2^r x - 2^{r-1} x^2) \sum_{n=0}^{\infty} P(r, n)x^n \\
 &= \sum_{n=0}^{\infty} P(r, n)x^n - 2^r \sum_{n=0}^{\infty} P(r, n)x^{n+1} - 2^{r-1} \sum_{n=0}^{\infty} P(r, n)x^{n+2} \\
 &= \sum_{n=2}^{\infty} (P(r, n) - 2^r P(r, n-1) - 2^{r-1} P(r, n-2))x^n \\
 &\quad + (P(r, 0) + P(r, 1)x) - 2^r P(r, 0)x
 \end{aligned}$$

By recurrence (2.1) we have

$$(1 - 2^r x - 2^{r-1} x^2)f(x) = 2 + (1 + 2^{r+1} - 2^{r+1})x.$$

Hence

$$(1 - 2^r x - 2^{r-1} x^2)f(x) = 2 + x.$$

Thus

$$f(x) = \frac{2 + x}{1 - 2^r x - 2^{r-1} x^2},$$

which ends the proof. □

#### 4. A graph interpretation of the $r$ -Pell numbers

In general we use the standard terminology and notation of graph theory, see [3]. Let  $G$  be a simple, undirected, finite graph with vertex set  $V(G)$  and edge set  $E(G)$ . By  $P_n$ ,  $C_m$ ,  $n \geq 1$ ,  $m \geq 3$ , we mean  $n$ -vertex path,  $m$ -vertex cycle, respectively. A set  $S \subseteq V(G)$  is independent if no edge of  $G$  has both its endpoints in  $S$ . Moreover, a subset of  $V(G)$  containing only one vertex and the empty set are independent sets of  $G$ . The total number of independent sets of a graph  $G$ , including the empty set, is known as the Merrifield-Simmons index. It is denoted by  $i(G)$  or  $NI(G)$ . For a graph  $G$  with  $V(G) = \emptyset$  we put

$i(G) = 1$ . The Merrifield-Simmons index is an example of topological index, which is of interest in combinatorial chemistry. This parameter was introduced in 1982 by Proding and Tichy in [7]. It was called the Fibonacci number of a graph. It has been proved that  $i(P_n) = F_{n+1}$ ,  $i(C_n) = L_n$ . In recent years, many researches have investigated this index, see for example [8]. We will show that the  $r$ -Pell numbers can be used for counting independent sets in special classes of graphs.

Let  $x \in V(G)$ . By  $i_x(G)$  ( $i_{-x}(G)$ , respectively) we denote the number of independent sets  $S$  of  $G$  such that  $x \in S$  ( $x \notin S$ , respectively). Hence we get the basic rule for counting of independent sets of a graph  $G$

$$(4.1) \quad i(G) = i_x(G) + i_{-x}(G).$$

Consider a graph  $H_{n,r}$  (Figure 1), where  $n \geq 1$ ,  $r \geq 1$ ,  $H_{1,r} = K_{1,r+1}$ .

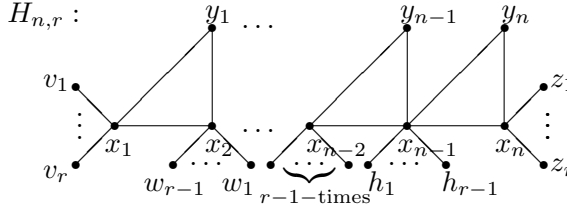


Figure 1. A graph  $H_{n,r}$

**THEOREM 4.1.** *Let  $n, r$  be integers such that  $n \geq 1$ ,  $r \geq 1$ . Then*

$$i(H_{n,r}) = P(r, n).$$

**PROOF.** Let  $n \geq 3$ . Assume that vertices of  $H_{n,r}$  are numbered as in Figure 1. Using formula (4.1), we have

$$i(H_{n,r}) = i_{x_n}(H_{n,r}) + i_{-x_n}(H_{n,r}).$$

Let  $S$  be any independent set of  $H_{n,r}$ . Consider two cases.

*Case 1.*  $x_n \in S$ . Then  $x_{n-1}, y_n, z_1, \dots, z_r \notin S$ . Hence  $S = S' \cup \{x_n\} \cup Z$ , where  $S'$  is any independent set of the graph

$$H_{n,r} \setminus \{x_{n-1}, y_n, z_1, \dots, z_r, h_1, \dots, h_r\},$$

which is isomorphic to  $H_{n-2,r}$ , and  $Z$  is any subset of the set  $\{h_1, h_2, \dots, h_{r-1}\}$ . Hence we get

$$i_{x_n}(H_{n,r}) = 2^{r-1} i(H_{n-2,r}).$$



Case 2.  $x_n \notin S$ . Proving analogously as in Case 1, we have

$$i_{-x_n}(H_{n,r}) = 2^r i(H_{n-2,r}).$$

Consequently, for  $n \geq 3$  we get

$$i(H_{n,r}) = 2^{r-1} i(H_{n-1,r}) + 2^r i(H_{n-2,r}).$$

Now we consider graphs  $H_{1,r}$  and  $H_{2,r}$ . It is easy to check that  $i(H_{1,r}) = 1 + 2^{r+1} = P(r, 1)$ . Using the same method for the graph  $H_{2,r}$  as in Case 1, we have

$$\begin{aligned} i(H_{2,r}) &= i_{x_2}(H_{2,r}) + i_{-x_2}(H_{2,r}) \\ &= 2^r + 2^r(1 + 2^{r+1}) = 2(4^r + 2^r) = P(r, 2). \quad \square \end{aligned}$$

COROLLARY 4.2. For  $n \geq 1$

$$i(H_{n,1}) = P(1, n) = P_{n+2}.$$

The graph interpretation of  $r$ -Pell numbers can be used for proving some identities.

THEOREM 4.3. (*Convolution identity*) Let  $n, m, r$  be integers such that  $m \geq 2, n \geq 1, r \geq 1$ . Then

$$P(r, m+n) = 2^{r-1} P(r, m-1) P(r, n) + 2^{2r-2} P(r, m-2) P(r, n-1).$$

PROOF. It is easy to check that the theorem is true for  $m = 2$  and  $n = 1$ , we have namely

$$P(r, 3) = 2^{r-1}(1 + 2^{r+1})^2 + 4 \cdot 2^{2r-2} = 2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r.$$

Moreover, for  $m = 2$  and  $n = 2$  we obtain

$$\begin{aligned} P(r, 4) &= 2^{r-1}(1 + 2^{r+1})(2^{r+1} + 2 \cdot 4^r) + 2^{2r-2}(2 + 2^{r+2}) \\ &= 2 \cdot 16^r + 4 \cdot 8^r + \frac{3}{2} \cdot 4^r. \end{aligned}$$

Assume now that  $m \geq 3, n \geq 2$ . Consider the graph  $H_{m+n,r}$ . Assume that vertices of the graph are numbered analogously as in Figure 1. By Theorem 4.1 we have  $i(H_{m+n,r}) = P(r, m+n)$ . Assume that  $x_m$  is any vertex of the graph  $H_{m+n,r}$ , such that  $\deg x_m = r+3$ . Let  $S$  be any independent set of the

graph  $H_{m+n,r}$ . Denote by  $L(x_i)$  the set of pendant vertices attached to the vertex  $x_i$ ,  $i = 1, 2, 3, \dots, m+n$ . Consider two cases.

*Case 1.*  $x_m \in S$ . Then  $x_{m-1}, x_{m+1}, y_m, y_{m-1} \notin S$ . Moreover,  $L(x_m) \not\subset S$ . Then  $S = S^* \cup S^{**} \cup Z_1 \cup Z_2 \cup \{x_m\}$ , where  $S^*$  is an independent set of the graph  $H_{m+n,r} \setminus \bigcup_{i=0}^{n+1} \{x_{m+n-i}\} \setminus \bigcup_{j=0}^{n+2} \{y_{m+n-j}\} \setminus L(x_i)$ , which is isomorphic to the graph  $H_{m-2,r}$ ,  $Z_1, Z_2$  is any subset of the set  $L(x_{m-1}), L(x_{m+1})$ , resp. Moreover,  $S^{**}$  is an independent set of the graph  $H_{m+n,r} \setminus \bigcup_{i=1}^{m+1} \{x_i, y_i\} \setminus L(x_i)$ , which is isomorphic to the graph  $H_{n-1,r}$ . Thus we obtain

$$i_{x_m}(H_{m+n,r}) = (2^{r-1})^2 P(r, m-2)P(r, n-1).$$

*Case 2.*  $x_m \notin S$ . Using the same method as in Case 1, we have

$$i_{-x_m}(H_{m+n,r}) = 2^{r-1} P(r, m-1)P(r, n).$$

Consequently,

$$\begin{aligned} i(H_{m+n,r}) &= P(r, m+n) \\ &= 2^{r-1} P(r, m-1)P(r, n) + 2^{2r-2} P(r, m-2)P(r, n-1). \quad \square \end{aligned}$$

Using the fact that  $P(0, n) = P_{n+2}$ , we get known identity for classical Pell numbers.

**COROLLARY 4.4.**  $P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$ .

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