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CATTANEO-VERNOTTE BIOHEAT TRANSFER EQUATION. STABILITY CONDITIONS OF NUMERICAL ALGORITHM BASED ON THE EXPLICIT SCHEME OF THE FINITE DIFFERENCE METHOD

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Abstract. The Cattaneo-Vernotte (CVE) equation is considered. This equation belongs to the group of hyperbolic PDE. Supplementing this equation by two additional terms corresponding to perfusion and metabolic heat sources one can apply the CVE as a mathematical model describing the heat transfer processes proceeding in domain of the soft tissue. Such an approach is recently often preferred substituting the classical Pennes model. At the stage of numerical computations the different numerical methods of the PDE solving can be used. In this paper the problems of stability conditions for the explicit scheme of the finite difference method (FDM) are discussed. The appropriate condition limiting the admissible time step have been found using the von Neumann analysis.

Keywords: *bioheat transfer, Cattaneo-Vernotte equation, finite difference method, stability conditions of FDM explicit scheme*

1. Introduction

Bioheat transfer processes proceeding in the domain of soft tissue are, as a rule, described using the well known Pennes equation [1-4] being a certain modification of the Fourier parabolic equation. As is well known the mathematical form of this equation, results from the assumption of the infinite velocity of thermal wave propagation. In the case of materials with a specific internal structure (e.g. biological tissue), the Fourier equation should be modified. To take into account the delay effect of the local and temporary heat flux with respect to the temperature gradient, the so-called relaxation time τ_q is introduced, and then the Cattaneo-Vernotte equation should be considered [5, 6]. According to literature, the relaxation time for the processed meat is the order of seconds (2÷5 s) [6, 7]. As mentioned, the bioheat transfer equation (the tissue model) contains two additional terms determining the perfusion and metabolic heat sources. The first is proportional to the local differences between blood and tissue temperatures. The second term (in this paper) is treated as a constant value. It should be pointed out, that the mathematical form of perfusion heat source results from the assumption that the tissue is supplied by the large number of blood capillaries uniformly distributed in the area under consideration. To take into account the presence of thermally significant vessels of considerable size the so-called vessels models are considered but these problems will not be discussed here.

The primary goal of this paper is to establish the stability conditions for the Cattaneo-Vernotte bioheat transfer equation (in the case when at the stage of numerical modeling, the explicit scheme of the finite difference method is used). The FDM equations are constructed in the version proposed in [8], while at the same time the 2D problem for domains oriented in the Cartesian co-ordinate system is considered.

2. The FDM equations

We consider the following energy equation

$$c\left[\frac{\partial T(x,y,t)}{\partial t}+\tau_q\frac{\partial^2 T(x,y,t)}{\partial t^2}\right]=\nabla\left[\lambda\nabla T(x,y,t)\right]+Q(x,y,t)+\tau_q\frac{\partial Q(x,y,t)}{\partial t} \tag{1}$$

where c is the volumetric specific heat of tissue, λ is the thermal conductivity, Q is the capacity of internal heat sources, τ_q is the relaxation time, T is the temperature, x, y, t denote the geometrical co-ordinates and time.
Additionally

$$Q(x,y,t)=G_Bc_B\left[T_B-T(x,y,t)\right]+Q_{met} \tag{2}$$

where G_B [m^3 blood/m^3 tissue/s] is the perfusion coefficient, c_B is the volumetric specific heat of blood, T_B is the arterial blood temperature, Q_{met} is the metabolic heat source (treated here as a constant value).

So, in the case of bio-heat problems the CVE is of the form

$$\begin{gathered}c\left[\frac{\partial T(x,y,t)}{\partial t}+\tau_q\frac{\partial^2 T(x,y,t)}{\partial t^2}\right]=\\ \nabla\left[\lambda\nabla T(x,y,t)\right]+G_Bc_B\left[T_B-T(x,y,t)\right]+Q_M(x,y,t)-\tau_qG_Bc_B\frac{\partial T(x,y,t)}{\partial t}\end{gathered} \tag{3}$$

The equation (2) is supplemented by the appropriate boundary and initial conditions. It should be pointed out that the form of typical boundary conditions in the case of CVE is somewhat different than the classic ones. For example, the Neumann condition takes a form

$$q(x,y,t)+\tau_q \frac{\partial q(x,y,t)}{\partial t}=-\lambda\nabla T(x,y,t) \tag{4}$$

One can see, that for the constant value of the boundary heat flux the condition (4) takes a well known form. The initial conditions determine the initial tissue temperature and initial cooling (heating) rate.

The numerical solution of the problem discussed can be obtained using the explicit scheme of the FDM. Let us consider the 2D differential mesh being the Cartesian product $\Omega_{h,k}\otimes\Omega_t$ of the geometrical mesh $\Omega_{h,k}$ (Fig. 1) and temporal one $\Omega_t:\left\{t^0,t^1,...,t^{s-2},t^{s-1},t^s,...t^S<\infty\right\}$. Both the geometric *h, k* and time Δt mesh steps are assumed to be the constant values.

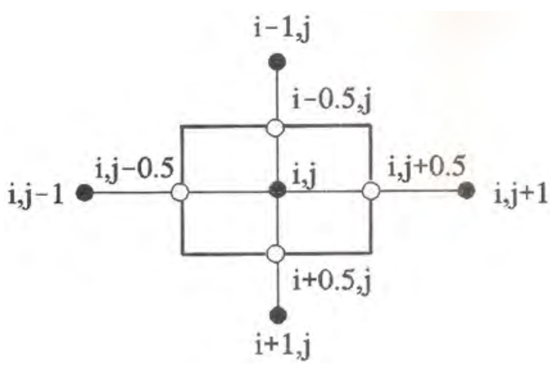

Fig. 1. Rectangular mesh

Below, the FDM equation for the internal nodes will be presented. To simplify the mathematical notation the local numbering of nodes is introduced, in particular the nodes (i, j), $(i, j+1)$, $(i, j-1)$, $(i+1, j)$, $(i-1, j)$ are denoted as 0, 1, 2, 3 and 4.

The FDM approximation of the Cattaneo-Vernotte equation can be taken in the following form

$$\begin{gathered} c\left(\frac{T_0^s-T_0^{s-1}}{\Delta t}+\tau_q\frac{T_0^s-2T_0^{s-1}+T_0^{s-2}}{(\Delta t)^2}\right)= \\ \sum_{e=1}^{4}\frac{T_e^{s-1}-T_0^{s-1}}{R_e^{s-1}}\Phi_e+G_Bc_B\left(T_B-T_0^{s-1}\right)+Q_M-\tau_qG_Bc_B\frac{T_0^s-T_0^{s-1}}{\Delta t}\end{gathered} \tag{5}$$

In the case of rectangular differental mesh

$$
\begin{aligned}
&\Phi_1=\Phi_2=\frac{1}{h}, \quad \Phi_3=\Phi_4=\frac{1}{k}\\
&R_1^{s-1}=\frac{0.5h}{\lambda_0^{s-1}}+\frac{0.5h}{\lambda_1^{s-1}}, \quad R_2^{s-1}=\frac{0.5h}{\lambda_0^{s-1}}+\frac{0.5h}{\lambda_2^{s-1}},\\
&R_3^{s-1}=\frac{0.5k}{\lambda_0^{s-1}}+\frac{0.5k}{\lambda_3^{s-1}}, \quad R_4^{s-1}=\frac{0.5k}{\lambda_0^{s-1}}+\frac{0.5k}{\lambda_4^{s-1}}
\end{aligned}
\tag{6}
$$

while Φ_e, R_e^{s-1} are the mesh shape functions and the thermal resistances between the neighboring nodes.

The equation (5) can be transformed as follows

$$
\begin{aligned}
&\frac{\Delta t\left(c+\tau_q G_B c_B\right)+\tau_q c}{\left(\Delta t\right)^2}T_0^s+\\
&\left[\frac{-\Delta t\left(c+\tau_q G_B c_B\right)-2\tau_q c+G_B c_B\left(\Delta t\right)^2}{\left(\Delta t\right)^2}+\sum_{e=1}^{4}\frac{\Phi_e}{R_e^{s-1}}\right]T_0^{s-1}+\frac{c\tau_q}{\left(\Delta t\right)^2}T_0^{s-2}=\\
&\sum_{e=1}^{4}\frac{\Phi_e}{R_e^{s-1}}T_e^{s-1}+G_B c_B T_B+Q_M
\end{aligned}
\tag{7}
$$

and finally

$$
\begin{aligned}
&T_0^s+\\
&\left[\frac{-\Delta t\left(c+\tau_q G_B c_B\right)-2\tau_q c+G_B c_B\left(\Delta t\right)^2}{\left(\Delta t\right)^2}+\sum_{e=1}^{4}\frac{\Phi_e}{R_e^{s-1}}\right]\frac{\left(\Delta t\right)^2}{\Delta t\left(c+\tau_q G_B c_B\right)+\tau_q c}T_0^{s-1}+\\
&\frac{c\tau_q}{\Delta t\left(c+\tau_q G_B c_B\right)+\tau_q c}T_0^{s-2}=\\
&\frac{\left(\Delta t\right)^2}{\Delta t\left(c+\tau_q G_B c_B\right)+\tau_q c}\sum_{e=1}^{4}\frac{\Phi_e}{R_e^{s-1}}T_e^{s-1}+\left(G_B c_B T_B+Q_M\right)\frac{\left(\Delta t\right)^2}{\Delta t\left(c+\tau_q G_B c_B\right)+\tau_q c}
\end{aligned}
\tag{8}
$$

Let us denote

$$
B_1=\left[\frac{-\Delta t\left(c+\tau_q G_B c_B\right)-2\tau_q c+G_B c_B\left(\Delta t\right)^2}{\left(\Delta t\right)^2}+\sum_{e=1}^{4}\frac{\Phi_e}{R_e^{s-1}}\right]\frac{\left(\Delta t\right)^2}{\Delta t\left(c+\tau_q G_B c_B\right)+\tau_q c}
\tag{9}
$$

$$B_2 = \frac{c\tau_q}{\Delta t\left(c + \tau_q G_B c_B\right) + \tau_q c} \tag{10}$$

$$A_e = \frac{\left(\Delta t\right)^2 \Phi_e}{\left[\Delta t\left(c + \tau_q G_B c_B\right) + \tau_q c\right] R_e^{s-1}} \tag{11}$$

$$D = \left(G_B c_B T_B + Q_M\right) \frac{\left(\Delta t\right)^2}{\Delta t\left(c + \tau_q G_B c_B\right) + \tau_q c} \tag{12}$$

and then

$$B T_0^s + B_1 T_0^{s-1} + B_2 T_0^{s-2} = \sum_{e=1}^{4} A_e T_e^{s-1} + D \tag{13}$$

3. Stability condition

The problem of numerical schemes stability is closely associated with a numerical error. The FDM scheme is stable when the errors made at one time step of the calculation do not cause the errors to increase as the computations are continued [9]. If, on the contrary, the errors grow with time the numerical scheme is said to be unstable. The stability of numerical schemes can be investigated by performing von Neumann stability analysis. According to this theory, the approximation error carried by θ_e^s at every node of space $(i, j) = (e)$ and time s is assumed to have a wave form with the wave numbers denoted by w_1, w_2 and the amplitude by δ:

$$\theta_e^s = \delta^s \exp\left[i\left(w_1 x_e + w_2 y_e\right)\right], \quad i = \sqrt{-1}, \qquad e = 0,1,\ldots,4 \tag{14}$$

As time progresses, to assure convergence, the amplitude of an approximation error must be less than unity, i.e. $\left|\theta_e^s\right| < 1$ [8-10].

Let us introduce the formula (14) into the FDM equation (13)

$$\begin{gathered}\delta^s \exp\left[i\left(w_1 x_0 + w_2 y_0\right)\right] + B_1 \delta^{s-1} \exp\left[i\left(w_1 x_0 + w_2 y_0\right)\right] + \\ B_2 \delta^{s-2} \exp\left[i\left(w_1 x_0 + w_2 y_0\right)\right] = \\ A_1 \delta^{s-1} \exp\left[i\left(w_1\left(x_0 + h\right) + w_2 y_0\right)\right] + A_2 \delta^{s-1} \exp\left[i\left(w_1\left(x_0 - h\right) + w_2 y_0\right)\right] + \\ A_3 \delta^{s-1} \exp\left[i\left(w_1 x_0 + w_2\left(y_0 + k\right)\right)\right] + A_4 \delta^{s-1} \exp\left[i\left(w_1 x_0 + w_2\left(y_0 - k\right)\right)\right] + D\end{gathered} \tag{15}$$

One can see, that for the rectagular mesh and the constant value of thermal conductivity $A_1 = A_2 = A_x$ and $A_3 = A_4 = A_y$.

Additionally the source term can be neglected, because it has no effect on the FDM equation stability. So

$$\begin{aligned}&\delta^s \exp\left[i\left(w_1 x_0 + w_2 y_0\right)\right] + B_1 \delta^{s-1} \exp\left[i\left(w_1 x_0 + w_2 y_0\right)\right] + \\ &B_2 \delta^{s-2} \exp\left[i\left(w_1 x_0 + w_2 y_0\right)\right] = \\ &A_x \delta^{s-1}\left\{\exp\left[i\left(w_1\left(x_0 + h\right) + w_2 y_0\right)\right] + \exp\left[i\left(w_1\left(x_0 - h\right) + w_2 y_0\right)\right]\right\} + \\ &A_y \delta^{s-1}\left\{\exp\left[i\left(w_1 x_0 + w_2\left(y_0 + k\right)\right)\right] + \exp\left[i\left(w_1 x_0 + w_2\left(y_0 - k\right)\right)\right]\right\}\end{aligned} \tag{16}$$

Dividing by δ^{s-2} one obtains

$$\begin{aligned}&\delta^2 + B_1\delta + B_2 = \\ &A_x \delta\left[\exp\left(i w_1 h\right) + \exp\left(-i w_1 h\right)\right] + A_y \delta\left[\exp\left(i w_2 k\right) + \exp\left(-i w_2 k\right)\right]\end{aligned} \tag{17}$$

or using the Euler formulas

$$\delta^2 + B_1\delta + B_2 = 2A_x \delta \cos\left(w_1 h\right) + 2A_y \delta \cos\left(w_2 k\right) \tag{18}$$

Denoting

$$D_1 = B_1 - 2A_x \cos\left(w_1 h\right) - 2A_y \cos\left(w_2 k\right), \quad D_2 = B_2 \tag{19}$$

one obtains the equation

$$\delta^2 + D_1\delta + D_2 = 0 \tag{20}$$

According to [9] the absolute values of the roots of equation (20) will be less than 1 when

$$\begin{aligned}&\left|D_2\right| < 1 \\ &\left|D_1\right| < 1 + D_2\end{aligned} \tag{21}$$

So, the first inequality takes a form

$$\left|\frac{c\tau_q}{\Delta t\left(c + \tau_q G_B c_B\right) + \tau_q c}\right| < 1 \tag{22}$$

From the last inequality one obtains

$$c\tau_q < \Delta t\left(c + \tau_q G_B c_B\right) + c\tau_q \tag{23}$$

or

$$\Delta t\left(c + \tau_q G_B c_B\right) > 0 \tag{24}$$

This inequality is unconditional and does not limit the time step.
Let us consider the second inequality, this means:

$$\left|B_1 - 2A_x \cos\left(w_1 h\right) - 2A_y \cos\left(w_2 k\right)\right| < 1 + \frac{c\tau_q}{\Delta t\left(c + \tau_q G_B c_B\right) + \tau_q c} \tag{25}$$

The left hand side of (25) can be trasformed in the following way

$$\begin{aligned} &\left|B_1 - 2A_x \cos\left(w_1 h\right) - 2A_y \cos\left(w_2 k\right)\right| = \\ &\left|B_1 - 2A_x + 2A_x\left[1 - \cos\left(w_1 h\right)\right] - 2A_y + 2A_y\left[1 - \cos\left(w_2 k\right)\right]\right| = \\ &\left|B_1 - 2A_x + 4A_x \sin^2\left(\frac{w_1 h}{2}\right) - 2A_y + 4A_y \sin^2\left(\frac{w_2 k}{2}\right)\right| \end{aligned} \tag{26}$$

From the view point of FDM equation stability the most ‘safe’ variant of the last inequality corresponds to $\sin^2\left(w_1 h / 2\right) = \sin^2\left(w_2 k / 2\right) = 1$ and then

$$\left|B_1 + 2A_x + 2A_y\right| < 1 + \frac{c\tau_q}{\Delta t\left(c + \tau_q G_B c_B\right) + \tau_q c} \tag{27}$$

For the constant value of thermal conductivity (see (6)) one obtains

$$0 < G_B c_B \left(\Delta t\right)^2 + 4\left(\Delta t\right)^2\left(\frac{\lambda}{h^2} + \frac{\lambda}{k^2}\right) < 2\Delta t\left(c + \tau_q G_B c_B\right) + 4\tau_q c \tag{28}$$

but the transition from (27) to (28) is very tedious.
The final form of CVE stability condition is the following

$$\left[G_B c_B + 4\left(\frac{\lambda}{h^2} + \frac{\lambda}{k^2}\right)\right]\left(\Delta t\right)^2 - 2\left(c + \tau_q G_B c_B\right)\Delta t - 4\tau_q c < 0 \tag{29}$$

In the case of non-linear tasks the stability condition can be also found. Then each FDM star for transition $t^{s-1} \to t^s$ must be considered individually and the critical time step corresponds to the lowest value, of course.

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