

THE JACOBIANS OF NON-MAXIMAL DEGREE

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Abstract. In the article the leading forms of the polynomial mapping having the Jacobians of non-maximal degree are considered. In particular, the mappings having two zeros at infinity are discussed.

Keywords: *Jacobian, zeros at infinity*

1. Introduction

The article presents the decomposition of leading forms of the polynomial mapping of two complex variables in the case when the Jacobian of this mapping does not meet maximal degree. In particular, the structure of these forms interest us in the case where the mapping has two zeros at infinity.

2. The Jacobian having non-generic degrees

Let f_m, h_n be the forms of variables X, Y of degrees m, n respectively with the condition $m > n \geq 2$. Suppose that the Jacobian of forms f_m, h_n vanish and represent the structures of these forms.

Lemma. *Let $\text{Jac}(f_m, h_n) = 0$. Therefore*

$$f_m = a \left[(\alpha_1 X + \beta_1 Y)^{p_1} \dots (\alpha_k X + \beta_k Y)^{p_k} \right]^{\tilde{m}} \quad (1)$$

and

$$h_n = b \left[(\alpha_1 X + \beta_1 Y)^{p_1} \dots (\alpha_k X + \beta_k Y)^{p_k} \right]^{\tilde{n}} \quad (2)$$

where:

$$\tilde{m} = m/(m, n), \quad \tilde{n} = n/(m, n) \quad \text{and} \quad p_1 + \dots + p_k = (m, n), \quad a, b, \alpha_i, \beta_i \in C,$$

$\det \begin{bmatrix} \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{bmatrix} \neq 0$ for $i \neq j$, where (m, n) means the greatest common divisor of the numbers m and n .

Proof.

According to the Euler formula [1] we have

$$X \frac{\partial f_m}{\partial X} + Y \frac{\partial f_m}{\partial Y} = m f_m \quad \text{and} \quad X \frac{\partial h_n}{\partial X} + Y \frac{\partial h_n}{\partial Y} = n h_n \quad (3)$$

so using the Cramer rule we obtain

$$X \text{Jac}(f_m, h_n) = \det \begin{bmatrix} m f_m & \frac{\partial f_m}{\partial Y} \\ n h_n & \frac{\partial h_n}{\partial Y} \end{bmatrix} \quad \text{and} \quad Y \text{Jac}(f_m, h_n) = \det \begin{bmatrix} \frac{\partial f_m}{\partial X} & m f_m \\ \frac{\partial h_n}{\partial X} & n h_n \end{bmatrix} \quad (4)$$

Because $\text{Jac}(f_m, h_n) = 0$, then

$$n \frac{\partial f_m}{\partial X} h_n = m \frac{\partial h_n}{\partial X} f_m \quad \text{and} \quad n \frac{\partial f_m}{\partial Y} h_n = m \frac{\partial h_n}{\partial Y} f_m \quad (5)$$

so after dividing by the greatest common divisor (m, n) of the numbers m and n we receive

$$\tilde{n} \frac{\partial f_m}{\partial X} = \tilde{m} \frac{\partial h_n}{\partial X} \quad \text{and} \quad \tilde{n} \frac{\partial f_m}{\partial Y} = \tilde{m} \frac{\partial h_n}{\partial Y} \quad (6)$$

This means that the logarithmic derivatives of the form f_m, h_n respectively, satisfy equalities

$$\left(\text{Log } f_m^{\tilde{n}} \right)'_X = \left(\text{Log } h_n^{\tilde{m}} \right)'_X \quad \text{and} \quad \left(\text{Log } f_m^{\tilde{n}} \right)'_Y = \left(\text{Log } h_n^{\tilde{m}} \right)'_Y \quad (7)$$

therefore

$$f_m^{\tilde{n}} = c h_n^{\tilde{m}} \quad (8)$$

Thus the forms f_m, h_n have the same factors.

So

$$f_m = a(\alpha_1 X + \beta_1 Y)^{\tilde{p}_1} \dots (\alpha_k X + \beta_k Y)^{\tilde{p}_k}, \quad \tilde{p}_1 + \dots + \tilde{p}_k = m \quad (9)$$

and

$$h_n = b(\alpha_1 X + \beta_1 Y)^{\tilde{q}_1} \dots (\alpha_k X + \beta_k Y)^{\tilde{q}_k}, \quad \tilde{q}_1 + \dots + \tilde{q}_k = n \quad (10)$$

According to formula (9) the exponents $\tilde{p}_1, \dots, \tilde{p}_k, \tilde{q}_1, \dots, \tilde{q}_k$ satisfy the equalities

$$\tilde{p}_1 \tilde{n} = \tilde{q}_1 \tilde{m}, \dots, \tilde{p}_k \tilde{n} = \tilde{q}_k \tilde{m} \quad (11)$$

Since the numbers \tilde{m}, \tilde{n} are relatively prime numbers, then

$$\tilde{m} | \tilde{p}_1, \dots, \tilde{m} | \tilde{p}_k \quad \text{and} \quad \tilde{n} | \tilde{q}_1, \dots, \tilde{n} | \tilde{q}_k \quad (12)$$

So

$$\tilde{p}_1 = p_1 \tilde{m}, \dots, \tilde{p}_k = p_k \tilde{m} \quad \text{and} \quad \tilde{q}_1 = q_1 \tilde{n}, \dots, \tilde{q}_k = q_k \tilde{n} \quad (13)$$

therefore

$$m = (p_1 + \dots + p_k) \tilde{m} \quad \text{and} \quad n = (q_1 + \dots + q_k) \tilde{n} \quad (14)$$

and at the same time the equality $p_1 + \dots + p_k = q_1 + \dots + q_k = (m, n)$ hold.

This means that

$$f_m = a \left[(\alpha_1 X + \beta_1 Y)^{p_1} \dots (\alpha_k X + \beta_k Y)^{p_k} \right]^{\tilde{m}} \quad (15)$$

and

$$h_n = b \left[(\alpha_1 X + \beta_1 Y)^{q_1} \dots (\alpha_k X + \beta_k Y)^{q_k} \right]^{\tilde{n}} \quad (16)$$

Assume

$$f_p = (\alpha_1 X + \beta_1 Y)^{p_1} \dots (\alpha_k X + \beta_k Y)^{p_k} \quad (17)$$

$$h_p = (\alpha_1 X + \beta_1 Y)^{q_1} \dots (\alpha_k X + \beta_k Y)^{q_k} \quad (18)$$

where $p = p_1 + \dots + p_k = q_1 + \dots + q_k$.

Therefore

$$f_m = a f_p^{\tilde{m}} \quad \text{and} \quad h_n = b h_p^{\tilde{n}} \quad (19)$$

so

$$\text{Jac}(f_m, h_n) = ab \tilde{m} \tilde{n} f_p^{\tilde{m}-1} h_p^{\tilde{n}-1} \text{Jac}(f_p, h_p) \quad (20)$$

Thus $\text{Jac}(f_m, h_n) = 0$ exactly when $\text{Jac}(f_p, h_p) = 0$. However

$$X \frac{\partial f_p}{\partial X} + Y \frac{\partial f_p}{\partial Y} = p f_p \quad \text{and} \quad X \frac{\partial h_p}{\partial X} + Y \frac{\partial h_p}{\partial Y} = p h_p \quad (21)$$

and

$$X \text{Jac}(f_p, h_p) = p \det \begin{bmatrix} f_p & \frac{\partial f_p}{\partial Y} \\ h_p & \frac{\partial h_p}{\partial Y} \end{bmatrix} \quad \text{and} \quad Y \text{Jac}(f_p, h_p) = p \det \begin{bmatrix} \frac{\partial f_p}{\partial X} & f_p \\ \frac{\partial h_p}{\partial X} & h_p \end{bmatrix} \quad (22)$$

Then

$$\frac{\partial f_p}{\partial X} h_p = \frac{\partial h_p}{\partial X} f_p \quad \text{and} \quad \frac{\partial f_p}{\partial Y} h_p = \frac{\partial h_p}{\partial Y} f_p \quad (23)$$

so

$$\frac{\frac{\partial f_p}{\partial X}}{f_p} = \frac{\frac{\partial h_p}{\partial X}}{h_p} \quad \text{and} \quad \frac{\frac{\partial f_p}{\partial Y}}{f_p} = \frac{\frac{\partial h_p}{\partial Y}}{h_p} \quad (24)$$

This means that

$$\left(\text{Log } f_p \right)'_X = \left(\text{Log } h_p \right)'_X \quad \text{and} \quad \left(\text{Log } f_p \right)'_Y = \left(\text{Log } h_p \right)'_Y \quad (25)$$

therefore

$$f_p = h_p = (\alpha_1 X + \beta_1 Y)^{p_1} \dots (\alpha_k X + \beta_k Y)^{p_k} \quad (26)$$

This completes the proof.

Remark 1. Obviously we can assume that $p_1 \geq \dots \geq p_k$.

Corollary 1 [2]. Let $f = f_m + \tilde{f}$, $h = h_n + \tilde{h}$, where $\det \tilde{f} < m$, $\det \tilde{h} < n$. If $\text{Jac}(f_m, h_n) = 0$, then only zeros at infinity of the mapping (f, h) are the factors of the form f_m or h_n .

Corollary 2. If the numbers m and n are relatively prime and $\text{Jac}(f_m, h_n) = 0$, then $f_m = a(\alpha X + \beta Y)^m$ and $h_n = b(\alpha X + \beta Y)^n$. This means that the mapping (f, h) has only one zero at infinity.

Corollary 3. Let $f = f_m + \tilde{f}$, $h = h_n + \tilde{h}$, where $\det \tilde{f} < m$ and $\det \tilde{h} < n$. Let $\text{Jac}(f_m, h_n) = 0$. If the mapping (f, h) have two zeros at infinity, then

$$f = (X^k Y^l)^p + \tilde{f} \quad \text{and} \quad h = (X^k Y^l)^q + \tilde{h} \quad (27)$$

where $k \geq l$, k and l are relatively prime, $p > q \geq 1$.

Remark 2. In particular, can be consider the case $f = X^{k+1} Y^{k+1} + \tilde{f}$ and $h = X^k Y^k + \tilde{h}$ (we put $k = l = 1$, $p = k + 1$, $q = k$). The mappings of this type were considered in the paper [3].

3. Conclusion

The Jacobians of non-maximal degree appears for the mappings with the constant Jacobian. The Jacobians Conjecture [4-9] do not occur for non trivial classes of the mappings having the constant Jacobian and one or two zeros in infinity [3, 10]. In the second case (two zeros in infinity) the leading forms of the mapping have the form given in the Corollary 2. The study of such mappings lead to the general classes of the mappings for which the Jacobian Conjecture does not take place.

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