

REPRESENTATIONS OF (D,O) -SPECIES AND FLAT MIXED MATRIX PROBLEMS

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Abstract. The problem of describing representations of (D,O) -species is reduced to flat mixed matrix problems over discrete valuation rings and their common skew field of fractions.

Keywords: *O-species, (D,O) -species, representations of (D,O) -species, (D,O) -species of bounded representation type, flat mixed matrix problem, discrete valuation ring*

1. Introduction

We continue the study of (D,O) -species that was started in [1]. These species generalize the notion of species introduced by Gabriel [2] and are the special kind of species considered in [3].

Let $\{O_i\}$ be a family of discrete valuation rings (not necessarily commutative) with a common skew field of fractions D . Consider a (D,O) -species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$, where $F_i = H_{n_i}(O_i)$ for $i = 1, 2, \dots, k$, and $F_j = D$ for $j = k+1, \dots, n$, moreover ${}_iM_j$ is an $(\tilde{F}_i, \tilde{F}_j)$ -bimodule that is finite dimensional both as the left D -vector space and as the right D -vector space, where \tilde{F}_i is a classical ring of fractions of F_i for $i = 1, 2, \dots, n$.

A (D,O) -species Ω is called **weak** if $F_i = O_i$ for all $i = 1, 2, \dots, k$, and moreover, ${}_iM_j = 0$ if $F_j = O_j$, and ${}_iM_j = {}_jM_i = 0$ for $i, j \in I$ and $i \neq j$.

For (D,O) -species the representations of O -species were defined in [1]. A representation $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ of a weak (D,O) -species $\Omega = \{F_i, {}_iM_j\}_{i,j \in I}$ is a family of right F_i -modules M_i ($i = 1, 2, \dots, k$), a set of right D -vector spaces V_r ($r = k+1, k+1, \dots, n$) and D -linear maps:

$${}_j\varphi_i : M_i \otimes_{F_i} {}_iM_j \rightarrow V_j$$

for each $i = 1, 2, \dots, k; j = k+1, k+2, \dots, n$; and

$${}_j\psi_r : V_r \otimes_D {}_rM_j \rightarrow V_j$$

for each $r, j = k+1, k+2, \dots, n$.

A representation V is said to be finite dimensional if all M_i are finitely generated F_i -modules and all V_r are finite dimensional D -vector spaces. A (D, O) -species is of bounded representation type if the dimensions (see (3.13) in [1]) of its indecomposable finite dimensional representations have an upper bound.

In this paper, we show that the description of representations of (D, O) -species can be reduced to some flat mixed matrix problems over discrete valuation rings and their common skew field of fractions. The definition of such matrix problems is given in Section 2. These matrix problems are some sort of generalization of a flat matrix problem considered by Zavadskii and Revitskaya [4]. Earlier such matrix problems were considered by Gubareni [5, 6], and Zavadskii and Kirichenko [7, 8]. Some examples of such flat matrix problems were also considered in [9]. The reduction of the problem of description of (D, O) -species of bounded representation type to some flat mixed matrix problems is given in Section 3.

With each weak (D, O) -species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ we can associate a D -species $\tilde{\Omega} = (\tilde{F}_i, {}_iM_j)_{i,j \in I}$, where $\tilde{F}_i = D$. In Section 4, we prove that if Ω is a simply connected weak (D, O) -species of bounded representation type, then $\tilde{\Omega}$ is a D -species of finite representation type.

2. Flat mixed matrix problems

Let O be a discrete valuation ring (DVR) with a classical division ring of fractions D . By left O -elementary transformations of rows of a matrix \mathbf{T} with entries in D we mean transformations of two types:

- a) multiplying a row on the left by an invertible element of O ;
- b) adding a row multiplied on the left by an element of O to another row.

In a similar way we can define left D -elementary transformations of rows and, by symmetry, right O -elementary and right D -elementary transformations of columns.

Elementary transformations (a) and (b) can be given by invertible elementary matrices. The automorphism of a finitely generated module P corresponding to an elementary transformation is an elementary automorphism. Multiplications on the left (right) side of a matrix \mathbf{T} by elementary matrices correspond to elementary row (column) transformations.

By [10, Proposition 13.1.3], any invertible matrix \mathbf{B} over a local ring O can be reduced by O -elementary row (column) transformations on \mathbf{B} to the identity matrix. By [10, Corollary 13.1.4], the matrix \mathbf{B} can be decomposed into a product of elementary matrices. Moreover, by [10, Theorem 13.1.6] any automorphism of a finitely

generated projective module P over a semiperfect ring A can be decomposed into a product of elementary automorphisms.

Let $\Delta = \{O_i\}_{i=1, \dots, k}$ be a family of discrete valuation rings O_i with a common skew field of fractions D . We define the general flat matrix problem over Δ and D in the following way.

Let

\mathbf{T}_{11}	\dots	\mathbf{T}_{1j}	\dots	\mathbf{T}_{1m}
\dots	\dots	\dots	\dots	\dots
\mathbf{T}_{i1}	\dots	\mathbf{T}_{ij}	\dots	\mathbf{T}_{im}
\dots	\dots	\dots	\dots	\dots
\mathbf{T}_{n1}	\dots	\mathbf{T}_{nj}	\dots	\mathbf{T}_{nm}

be a block rectangular matrix \mathbf{T} with entries in D partitioned into n horizontal strips $\mathbf{T}_1, \dots, \mathbf{T}_n$ and m vertical strips $\mathbf{T}^1, \dots, \mathbf{T}^m$ so that each block \mathbf{T}_{ij} is the intersection of the j -th vertical strip and the i -th horizontal strip; some of these blocks may be empty.

Assume that the ring $F_{i_s} \in \Delta \cup D$ corresponds to the i -th horizontal strip \mathbf{T}_i and the ring $F_{j_t} \in \Delta \cup D$ corresponds to the j -th vertical strip \mathbf{T}^j .

The following transformations with the matrix \mathbf{T} are admissible:

1. Left F_{i_s} -elementary transformations of rows within the strip \mathbf{T}_i .
2. Right F_{j_t} -elementary transformations of rows within the strip \mathbf{T}^j .
3. Additions of rows in the strip \mathbf{T}_j multiplied on the left by elements of $F_r \in \Delta \cup D$ to rows in the strip \mathbf{T}_i .
4. Additions of columns in the strip \mathbf{T}^i multiplied on the right by elements of $F_p \in \Delta \cup D$ to columns in the strip \mathbf{T}^j .

Indecomposable matrices and equivalent matrices are defined in a natural way.

A flat matrix problem is said to be of **finite type** if the number non-equivalent indecomposable matrices is finite.

Definition 2.1. The vector

$$\mathbf{d} = \mathbf{d}(\mathbf{T}) = (d_1, d_2, \dots, d_n; d^1, d^2, \dots, d^m), \quad (2.2)$$

where d_i is the number of rows of the i -th horizontal strip of \mathbf{T} for $i = 1, \dots, n$ and d^j is the number of columns of the j -th vertical strip of \mathbf{T} for $j = 1, \dots, m$, is called the **dimension vector** of the partition matrix \mathbf{T} . Also set

$$\dim(\mathbf{T}) = \sum_{i=1}^n d_i + \sum_{j=1}^m d^j \quad (2.3)$$

Definition 2.4.

A flat matrix problem is said to be of **bounded representation type** if there is a constant C such that $\dim(\mathbf{X}) < C$ for all indecomposable matrices \mathbf{X} . Otherwise it is of **unbounded representation type**.

3. The main matrix problem

Let $\Omega = (F_i, {}_iM_j)_{i,j \in I}$, where $F_i = O_i$ for $i = 1, 2, \dots, k$ and $F_j = D$ for $j = k+1, \dots, n$, be a weak (D, O) -species of bounded representation type.

Suppose that $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ is an indecomposable finite dimensional representation of Ω . Then M_i is a finitely generated F_i -module for $i = 1, 2, \dots, k$ and V_r is a finite dimensional D -vector space for $r = k+1, \dots, n$. Since $F_i = O_i$ is a discrete valuation ring, by [3, Proposition 5.4.18], any O_i -module M_i is torsion-free and faithful. Therefore any indecomposable representation of Ω has the following form:

$$V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r) \quad (3.1)$$

where M_i is a free F_i -module.

Consider the category $R(\Omega)$ whose objects are representations $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$, and a morphism from an object V to an object $V' = (M'_i, V'_r, {}_j\varphi'_i, {}_j\psi'_r)$ is a set of homomorphisms (α_i, β_r) , in which $\alpha_i: M_i \rightarrow M'_i$ is a homomorphism of F_i -modules, $\beta_r: V_r \rightarrow V'_r$ is a homomorphism of D -vector spaces ($r = k+1, \dots, n$), and the following equalities hold:

$${}_j\varphi'_i(\alpha_i \otimes 1) = \beta_j \cdot {}_j\varphi_i \quad (3.2)$$

$${}_j\psi'_r(\beta_r \otimes 1) = \beta_j \cdot {}_j\psi_r \quad (3.3)$$

Let V be an indecomposable finite dimensional representation of the (D, O) -species Ω . Thus, each M_i is a finitely generated free O_i -module with basis $\omega_1^{(i)}, \dots, \omega_{m_i}^{(i)}$ ($i = 1, 2, \dots, k$); and V_r is a finite dimensional D -space with basis $\tau_1^{(r)}, \dots, \tau_{k_r}^{(r)}$ ($r = k+1, \dots, n$).

Suppose

$${}_j\varphi_i(\omega_s^{(i)} \otimes 1) = \sum_{u=1}^{k_j} \tau_u^{(j)} b_{us}^{(ij)} \quad (3.4)$$

$${}_j\psi_r(\tau_v^{(r)} \otimes 1) = \sum_{u=1}^{k_j} \tau_u^{(j)} a_{uv}^{(ij)} \quad (3.5)$$

Then the matrices $\mathbf{A}_{ij} = (a_{uv}^{(ij)})$, $\mathbf{B}_{ij} = (b_{us}^{(ij)})$ define the representation V uniquely up to equivalence.

Let $\mathbf{U}_i \in M_{m_i}(F_i)$ be the matrix corresponding to the homomorphism α_i , and let $\mathbf{W}_i \in M_{k_i}(D)$ be the matrices corresponding to the homomorphisms β_i , $i \in I$. If \mathbf{A}'_{ij} , \mathbf{B}'_{ij} are the matrices corresponding to a representation V' then the equalities (3.2) and (3.3) have the following matrix form:

$$\mathbf{W}_i \mathbf{B}_{ij} = \mathbf{B}'_{ij} \mathbf{U}_j \quad (i=1, \dots, k; j=k+1, \dots, n) \quad (3.6)$$

$$\mathbf{W}_j \mathbf{A}_{jr} = \mathbf{A}'_{jr} \mathbf{W}_r \quad (j, r=k+1, \dots, n) \quad (3.7)$$

If representations V and V' are equivalent, then α_i, β_r are isomorphisms. Therefore, the matrices \mathbf{U}_i and \mathbf{W}_r are invertible and the equalities (3.2) and (3.3) are equivalent to the following equalities:

$$\mathbf{W}_i \mathbf{B}_{ij} \mathbf{U}_j^{-1} = \mathbf{B}'_{ij} \quad (i=1, \dots, k; j=k+1, \dots, n) \quad (3.8)$$

$$\mathbf{W}_j \mathbf{A}_{jr} \mathbf{W}_r^{-1} = \mathbf{A}'_{jr} \quad (j, r=k+1, \dots, n) \quad (3.9)$$

Thus we obtain the following matrix problem for description of indecomposable finite dimensional representations of a (D, O) -species Ω .

Main mixed matrix problem

Let $\Delta = \{O_i\}_{i=1,2,\dots,k}$ be a family of discrete valuation rings O_i with a common skew field of fractions D .

Let \mathbf{T} be a block matrix with entries in D partitioned into n horizontal strips $\{\mathbf{T}_i\}_{i=1,\dots,n}$ and m vertical strips $\{\mathbf{T}^j\}_{j=1,\dots,m}$ so that each block \mathbf{T}_{ij} is the intersection of j -th vertical strip and i -th horizontal strip, some of these matrices may be empty.

The following transformations with the matrix \mathbf{T} are admissible:

1. Left F_{i_s} -elementary transformations of rows within the strip \mathbf{T}_i , where $F_{i_s} \in \Delta \cup D$.
2. Right F_{j_t} -elementary transformations of rows within the strip \mathbf{T}^j , where $F_{j_t} \in \Delta \cup D$.

The admissible transformations with the matrix \mathbf{T} can be given in the form $\mathbf{T} \rightarrow \mathbf{XTY}$, where $\mathbf{X} = \text{diag}(\mathbf{X}_1, \dots, \mathbf{X}_n)$ and $\mathbf{Y} = \text{diag}(\mathbf{Y}_1, \dots, \mathbf{Y}_m)$, and all \mathbf{X}_i and \mathbf{Y}_j are square invertible matrices. Moreover, $\mathbf{X}_i \subset M_{m_i}(F_{i_s})$, and $\mathbf{Y}_j \subset M_{k_j}(F_{j_t})$, where $F_{i_s}, F_{j_t} \in \Delta \cup D$.

Clearly, the matrix \mathbf{T} is indecomposable if and only if the corresponding representation of Ω is indecomposable. It is easy to prove the following statement.

Lemma 3.10. *A (D, O) -species Ω is of bounded representation type if and only if the corresponding main matrix problem is of bounded representation type.*

4. Weak (D, O) -species of bounded representation type

Let $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ be O -species. The **quiver** $\Gamma(\Omega)$ of an O -species Ω is defined as the directed graph whose vertices are $1, \dots, n$, and there is an arrow from the vertex i to the vertex j if and only if ${}_iM_j \neq 0$.

An O -species Ω is called **acyclic** if its quiver has no oriented cycles, i.e. the indices can be chosen so that ${}_iM_i = 0$ for all i , and ${}_iM_j = 0$ for $j \leq i$.

A vertex $i \in I$ is called marked if $F_i = H_{n_i}(O_i)$. Let $I_1 = \{1, 2, \dots, k\}$ be the set of marked vertices of an O -species Ω . A marked vertex $i \in I_1$ is called **minimal** if ${}_iM_j = {}_jM_i = 0$ for all $j \in I_1$. An O -species Ω is called **min-marked** if all its marked vertices are minimal.

An O -species Ω is **simply connected** if the underlying graph of $\Gamma(\Omega)$ is a tree.

A (D, O) -species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ is said to be weak if Ω is min-marked and all F_i are O_i or D .

For each O -species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ in [1], the tensor algebra $T(\Omega) = \bigoplus_{i=0}^{\infty} T_i$, where $T_0 = \prod_{i=1}^n F_i = B$, $T_{i+1} = T_i \otimes_B M$ and $M = \bigoplus_{i,j=1}^{\infty} {}_iM_j$, was constructed.

Lemma 4.1. *Let $\Omega = (F_i, {}_iM_j)_{i,j \in I}$, where all $F_i = D$, be a simply connected D -species of finite representation type. Then the tensor algebra $T(\Omega)$ is a hereditary Artinian semidistributive ring.*

Proof. Since Ω is a simply connected species, the tensor algebra $T(\Omega)$ is Morita equivalent to the algebra

$$A = \begin{pmatrix} D & & A_{ij} \\ & \ddots & \\ 0 & & D \end{pmatrix}$$

where

$$A_{ij} = \bigoplus_{i=i_0 < i_1 < \dots < i_k = j} {}_{i_0}M_{i_1} \otimes_{{}_{i_1}} M_{i_2} \otimes \dots \otimes_{{}_{i_{k-1}}} M_{i_k} \quad (4.2)$$

Since all ${}_i M_j$ are finitely dimensional right and left D -spaces, A is an Artinian ring. From [11, Corollary 2.2.13] it follows that A is a hereditary ring.

Note that the ring

$$\begin{pmatrix} D & V_{12} \\ 0 & D \end{pmatrix} \quad (4.3)$$

where V_{12} is a (D, D) -bimodule, is of finite representation type if and only if V_{12} has dimension 1 both as right and as left D -vector space. Since Ω is a D -species of finite representation type, the tensor algebra $T(\Omega)$ is of finite representation type as well, and so it does not contain a minor that is isomorphic to the ring (4.3). Therefore, A is a semidistributive ring.

Besides a weak (D, O) -species $\Omega = (F_i, {}_i M_j)_{i,j \in I}$ we can also consider a D -species $\tilde{\Omega} = (\tilde{F}_i, {}_i \tilde{M}_j)_{i,j \in I}$, where $\tilde{F}_i = D$, since each ${}_i M_j$ is an $(\tilde{F}_i, \tilde{F}_j)$ -bimodule. Let $T(\tilde{\Omega})$ be a tensor algebra of D -species $\tilde{\Omega}$. Since $T(\tilde{\Omega})$ is an Artinian ring, by [12, 13] it is of bounded representation type if and only if it is of finite representation type.

Proposition 4.4. *If Ω is a weak simply connected (D, O) -species of bounded representation type, then $\tilde{\Omega}$ is a D -species of finite representation type.*

Proof. Let Ω be a weak simply connected (D, O) -species with set of marked vertices $J = \{1, 2, \dots, k\}$. Then the tensor algebra $A = T(\Omega)$ is a basic primely triangular ring whose two-sided Peirce decomposition has the following form

$$A = T(\Omega) = \begin{pmatrix} O_1 & \cdots & 0 & U_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & O_k & U_k \\ 0 & \cdots & 0 & T \end{pmatrix} \quad (4.4)$$

where each U_i is a (D, T) -bimodule. Moreover, the ring T is the tensor algebra of a species $\Omega_1 = (F_i, {}_i M_j)_{i,j \in I \setminus J}$, where $F_i = D$ for all $i \in I \setminus J$.

Since Ω is a (D, O) -species of bounded representation type, then the tensor algebra $T(\Omega)$ is also of bounded representation type by [1, Corollary 3.15]. Then by [1, Corollary 3.16], T is also of bounded representation type. Since Ω_1 is a D -species, T is an Artinian ring and so it is of finite representation type. Since Ω is simply connected, Ω_1 is also simply connected. By Lemma 4.1, T is an Artinian hereditary semidistributive ring.

Let \tilde{A} be a right classical ring of fractions of A . We will use the following notation: if M is a right A -module, then $M' = M \otimes_A \tilde{A}$; and if M is a right \tilde{A} -module,

then M' is the module M considered as an A -module. The length of a composition series of a right \tilde{A} -module X is denoted by $l(X)$.

Let us prove that for any right \tilde{A} -module M there is a right \tilde{A} -module X such that $M'' = M \oplus X$.

We have

$$M'' = M'' \otimes_A \tilde{A} = (M \otimes_A \tilde{A}) \otimes_A \tilde{A}.$$

Taking into account (4.5), we have that $M = \bigoplus_{i=1}^k M_i \oplus M_0$, where M_i is an O_i -module and M_0 is a T -module. Then

$$M \otimes_A \tilde{A} = \left(\bigoplus_{i=1}^k M_i \oplus M_0 \right) \otimes_A \tilde{A} = \bigoplus_{i=1}^k (M_i \otimes_{O_i} D) \oplus M_0$$

$$M'' = (M \otimes_A \tilde{A}) \otimes_A \tilde{A} = \bigoplus_{i=1}^k M_i \otimes_{O_i} (D \otimes_{O_i} D) \oplus M_0.$$

By [14, Lemma 2], there is an injective torsion-free O_i -module for each $i = 1, \dots, k$. Therefore, the mapping $D \rightarrow D \otimes_{O_i} D$ with $d \mapsto 1 \otimes d$ for each $d \in D$ is a monomorphism, i.e. exact sequences of O_i -modules exist:

$$0 \rightarrow D \rightarrow D \otimes_{O_i} D \rightarrow \text{Coker}(\varphi_i) \rightarrow 0$$

Since D is injective, these sequences split, i.e. $D \otimes_{O_i} D = D \oplus Y_i$ for $i = 1, \dots, k$. Therefore,

$$\begin{aligned} M'' &= \bigoplus_{i=1}^k M_i \otimes_{O_i} (D \oplus Y_i) \oplus M_0 = \bigoplus_{i=1}^k ((M_i \otimes_{O_i} D) \oplus (M_i \otimes_{O_i} Y_i)) \oplus M_0 = \\ &= M \oplus X. \end{aligned}$$

Now suppose that the ring A is of bounded representation type and the ring \tilde{A} is of infinite representation type. Then for any $N > 0$ there is an indecomposable finitely generated \tilde{A} -module M such that $l(M) > N$.

Consider the A -module M' . It is finitely generated and, by [15, Proposition 1], it decomposes into a direct sum of finitely generated indecomposable A -modules:

$$M' = N_1 \oplus \dots \oplus N_t.$$

Then

$$M'' = N'_1 \oplus \dots \oplus N'_t$$

Since $M'' = M \oplus X$, and M'' is a finitely generated module over an Artinian ring \tilde{A} , it follows from the uniqueness of the decomposition that there is a number i such that M is a direct summand of N'_i , i.e. there is an \tilde{A} -module P such that $N'_i = M \oplus P$. We have the chain of inequalities

$$\mu_A(N_i) = \mu_{\tilde{A}}(N'_i) \geq l(N'_i) = l(M) + l(P) \geq l(M) > N,$$

which contradicts the assumption that A is of bounded representation type.

5. Conclusions

The problem of describing representations of (D, O) -species has been reduced to some flat matrix problems over discrete valuation rings with common skew field of fractions. The main matrix problem for description of (D, O) -species of bounded representation type is given. We establish the connection of (D, O) -species of bounded representation type with D -species of finite representation type. We prove that if Ω is a weak simply connected (D, O) -species of bounded representation type, then the corresponding D -species $\tilde{\Omega}$ is of finite representation type.

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