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A SECOND EXAMPLE OF NON-KELLER MAPPING

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Abstract. In the article the next nontrivial example of non-Keller mapping having two zeros at infinity is analyzed. The rare mapping of two complex variables having two zeros at infinity is considered. In the article it has been proved that if the Jacobian of the considered mapping is constant, then it is zero.

Keywords: *Jacobian, zeros at infinity, rare mappings, Keller mapping*

1. Introduction

In this article we analyze the rare polynomial mappings of two complex variables. We consider the mappings having two zeros at infinity [1-3]. It has been shown that if the Jacobian of such mappings is constant, it must be zero. The work is related to the Keller mapping [4-6] (the Keller mapping is a polynomial mapping $F:\mathbb{C}^2 \to \mathbb{C}^2$ with the condition $\operatorname{Jac} F = \text{const} \neq 0$). In the presented paper, the non-Keller mappings are those for which the Jacobian, if it is constant, is zero.

2. The rare mappings

Let $f, h:\mathbb{C}^2 \to \mathbb{C}$ are the complex polynomials of degrees $4k+2$ and $4k$, consequently, and having two zeros at infinity. Assume

$$f = X^{2k+1}Y^{2k+1} + \underbrace{0 + \ldots + 0}_{2k-1 \text{ zeros}} + f_{2k+2} + f_{2k+1} + f_{2k} + \ldots + f_3 + f_2 + f_1 \tag{1}$$

and

$$h = X^{2k}Y^{2k} + \underbrace{0 + \ldots + 0}_{2k-1 \text{ zeros}} + h_{2k} + h_{2k-1} + h_{2k-2} + \ldots + h_1 \tag{2}$$

where $k \geq 2$ and $f_i, h_j \in \mathbf{C}[X, Y]$ are the forms of the indicated degrees. These mappings are called rare. Suppose

$$\text{Jac}(f, h) = \text{Jac}(f_1, h_1) = \text{const} \tag{3}$$

Let's prove that $\text{Jac}(f, h) = 0$.

3. Basic lemma

Let us provide the following property [7]:

Property. *If* $X^{2k-1}Y^{2k-1} \big| h_{2k}^2$ *, then* $h_{2k} = B_{2k} X^{2k} \mathrm{Y}^{2k}, \quad k \geq 1$.

Lemma. With the given assumptions we have $\text{Jac}(f, h) = 0$.
Proof.
Let

$$\begin{aligned} f = X^{2k+1}Y^{2k+1} + \underbrace{{}^{2k+3)}\big|0 + \ldots + 0}_{2k-1 \text{ zeros}} + f_{2k+2}\Big|^{1)} + f_{2k+1}\Big|^{2)} + f_{2k}\Big|^{3)} + \ldots + \\ + f_3\Big|^{2k)} + f_2\Big|^{2k+1)} + f_1\Big|^{2k+2)} \end{aligned} \tag{4}$$

$$\begin{aligned} h = X^{2k}Y^{2k} + \underbrace{0 + \ldots + {}^{2k+3)}\big|0}_{2k-1 \text{ zeros}} + h_{2k}\Big|^{1)} + h_{2k-1}\Big|^{2)} + h_{2k-2}\Big|^{3)} + \ldots + \\ + h_1\Big|^{2k)} + 0\Big|^{2k+1)} + 0\Big|^{2k+2)} \end{aligned} \tag{5}$$

Since the Jacobian is constant, we have consecutively

$$1) \quad \text{Jac}\left(X^{2k+1}Y^{2k+1}, h_{2k}\right) = \text{Jac}\left(X^{2k}Y^{2k+1}, f_{2k+2}\right) \tag{6}$$

so

$$\frac{2k+1}{2k} XY h_{2k} + a_{2k+2} X^{k+1} Y^{k+1} = f_{2k+2} \tag{7}$$

and next

$$2) \quad \text{Jac}\left(X^{2k+1}Y^{2k+1}, h_{2k-1}\right) = \text{Jac}\left(X^{2k}Y^{2k}, f_{2k+1}\right) \tag{8}$$

so

$$\frac{2k+1}{2k} X\,Y\,h_{2k-1} = f_{2k+1} \tag{9}$$

and

$$3)\quad \operatorname{Jac}\left(X^{2k+1}Y^{2k+1}, h_{2k-2}\right) = \operatorname{Jac}\left(X^{2k}Y^{2k}, f_{2k}\right) \tag{10}$$

then

$$\frac{2k+1}{2k} X\,Y\,h_{2k-2} + a_{2k} X^{k} Y^{k} = f_{2k} \tag{11}$$

etc.
In the $2k$-step we have

$$2k)\quad \operatorname{Jac}\left(X^{2k+1}Y^{2k+1}, h_{1}\right) = \operatorname{Jac}\left(X^{2k}Y^{2k}, f_{3}\right) \tag{12}$$

so

$$\frac{2k+1}{2k} X\,Y\,h_{1} = f_{3} \tag{13}$$

In the next step we obtain

$$2k+1)\quad \underbrace{\operatorname{Jac}\left(f_{2k+2}, h_{2k}\right)}_{1^\circ} = \operatorname{Jac}\left(X^{2k}Y^{2k}, f_{2}\right) \tag{14}$$

where

$$\begin{aligned} 1^\circ &= \operatorname{Jac}\left(f_{2k+2}, h_{2k}\right) = \operatorname{Jac}\left(\frac{2k+1}{2k} X\,Y\,h_{2k} + a_{2k+2} X^{k+1} Y^{k+1}, h_{2k}\right) = \\ &= \frac{2k+1}{2k} \operatorname{Jac}\left(X\,Y\,h_{2k}, h_{2k}\right) + a_{2k+2} \operatorname{Jac}\left(X^{k+1}Y^{k+1}, h_{2k}\right) = \\ &= \frac{2k+1}{2k} h_{2k} \operatorname{Jac}\left(X\,Y, h_{2k}\right) + (k+1) a_{2k+2} X^{k} Y^{k} \operatorname{Jac}\left(X\,Y, h_{2k}\right) \end{aligned} \tag{15}$$

and taking into account the formula (14), we have

$$\begin{aligned} &\frac{2k+1}{2k} h_{2k} \operatorname{Jac}\left(X\,Y, h_{2k}\right) + (k+1) a_{2k+2} X^{k} Y^{k} \operatorname{Jac}\left(X\,Y, h_{2k}\right) = \\ &= 2k\,X^{2k-1}Y^{2k-1} \operatorname{Jac}\left(X\,Y, f_{2}\right) \end{aligned} \tag{16}$$

so

$$\frac{2k+1}{4k}h_{2k}^2 + (k+1)a_{2k+2}X^kY^kh_{2k} + b_{4k}X^{2k}Y^{2k} = 2kX^{2k-1}Y^{2k-1}f_2 \tag{17}$$

Thus $X^{2k-1}Y^{2k-1}$ divides h_{2k}^2 (see Property), therefore

$$h_{2k} = B_{2k}X^kY^k, \qquad f_{2k+2} = A_{2k+2}X^{k+1}Y^{k+1} \tag{18}$$

and

$$f_2 = A_2 XY \tag{19}$$

In $2k+2$-step we get

$$(2k+2)\,\mathrm{Jac}\left(f_{2k+2}, h_{2k-1}\right) + \mathrm{Jac}\left(f_{2k+1}, h_{2k}\right) = \mathrm{Jac}\left(X^{2k}Y^{2k}, f_1\right) \tag{20}$$

Returning to the formulas (18) and (9), we have

$$\begin{aligned}&(k+1)A_{2k+2}\,X^kY^k\,\mathrm{Jac}\left(XY, h_{2k-1}\right) - \frac{2k+1}{2k}B_{2k}\,X^kY^k\,\mathrm{Jac}\left(XY, h_{2k-1}\right) = \\ &= 2kX^{2k-1}Y^{2k-1}\,\mathrm{Jac}\left(XY, f_1\right)\end{aligned} \tag{21}$$

hence

$$\frac{1}{2k}C_k\,\mathrm{Jac}\left(XY, h_{2k-1}\right) = X^{2k-1}Y^{2k-1}\,\mathrm{Jac}\left(XY, f_1\right) \tag{22}$$

and so

$$\frac{1}{2k}C_k\,h_{2k-1} = X^{k-1}Y^{k-1}f_1 \tag{23}$$

where

$$C_k = (k+1)A_{2k+2} - \frac{2k+1}{2k}B_{2k} \tag{24}$$

Therefore $X^{k-1}Y^{k-1}\big|h_{2k-1}$, thus

$$h_{2k-1} = X^{k-1}Y^{k-1}h_{2k-1|1}, \qquad f_{2k+1} = \frac{2k+1}{2k}X^kY^k\,h_{2k-1|1} \tag{25}$$

and

$$h_{2k-1} = X^{k-1} Y^{k-1} h_{2k-1|1}, \qquad f_1 = \frac{1}{2k} C_k h_{2k-1|1} \tag{26}$$

In the next step we obtain

$$2k+3) \quad \mathrm{Jac}\left(f_{2k+2}, h_{2k-2}\right) + \underbrace{\mathrm{Jac}\left(f_{2k+1}, h_{2k-1}\right)}_{1^\circ} + \mathrm{Jac}\left(f_{2k}, h_{2k}\right) = 0 \tag{27}$$

where

$$\begin{aligned} 1^\circ &= \mathrm{Jac}\left(\frac{2k+1}{2k} X^k Y^k h_{2k-1|1},\ X^{k-1} Y^{k-1} h_{2k-1|1}\right) = \\ &= \frac{2k+1}{2k} X^{2k-2} Y^{2k-2} h_{2k-1|1}\ \mathrm{Jac}\left(XY, h_{2k-1|1}\right) \end{aligned} \tag{28}$$

Back to formula (27) we get

$$C_k h_{2k-2} + \frac{2k+1}{4k} X^{2k-1} Y^{2k-1} h_{2k-1|1}^2 + \widehat{b}_{2k-2} X^{k-1} Y^{k-1} = 0 \tag{29}$$

Therefore

$$h_{2k-2} = -\frac{2k+1}{4k} \frac{1}{C_k} X^{k-2} Y^{k-2} h_{2k-1|1}^2 + \tilde{b}_{2k-2} X^{k-1} Y^{k-1} \tag{30}$$

and recalling formula (11) we receive

$$f_{2k} = -\frac{(2k+1)^2}{8k^2} \frac{1}{C_k} X^{k-1} Y^{k-1} h_{2k-1|1}^2 + \tilde{a}_{2k} X^k Y^k \tag{31}$$

In the following steps to reduce the power of variables (one with every step). The odd steps are an even power, and even steps are the odd power of the monomial $h_{2k-1|1}$. In the step (3*k* + 2), the largest power $h_{2k-1|1}$ appears, namely $h_{2k-1|1}^{k+1}$. Then, $XY \mid h_{2k-1|1}$ and this means that $h_{2k-1|1} = 0$. Hence $f_1 = 0$ (equation (26)), so $\mathrm{Jac}\left(f_1, h_1\right) = 0$. Which completes the proof of the lemma.

4. Conclusion

In the considered example, the form $h_{2k-1|1}$ was essential. If we considered the case

$$f = X^{2k+1}Y^{2k+1} + \underbrace{0 + \ldots + 0}_{2k-2 \text{ zeros}} + f_{2k+3} + f_{2k+2} + f_{2k+1} + \ldots + f_3 + f_2 + f_1 \quad (32)$$

and

$$h = X^{2k}Y^{2k} + \underbrace{0 + \ldots + 0}_{2k-2 \text{ zeros}} + h_{2k+1} + h_{2k} + h_{2k-1} + \ldots + h_1 \quad (33)$$

where $2k - 2$ appears, then difficult and more interesting considerations show that the above case depends on the form $h_{2k+1|1}$. In this paper, the presented case of rare mapping is therefore a "frontier" case, which is rare and non-Keller mapping having two zeros at infinity. Some remarks on the general case

$$f = X^{k+1}Y^{k+1} + f_{2k+1} + f_{2k} + f_{2k-1} + \ldots + f_3 + f_2 + f_1 \quad (34)$$

and

$$h = X^{k}Y^{k} + h_{2k-1} + h_{2k-2} + h_{2k-3} + \ldots + h_1 \quad (35)$$

will be presented in the later articles.

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