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O-SPECIES AND TENSOR ALGEBRAS

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Abstract. In this paper we consider *O*-species and their representations. These *O*-species are a type of a generalization of a species introduced by Gabriel. We also consider the tensor algebras of such *O*-species. It is proved that the category of all representations of an *O*-species and the category of all right modules over the corresponding tensor algebra are naturally equivalent.

Keywords: *species, O-species, representations of O-species, tensor algebra, O-species of bounded representation type, diagram of O-species*

1. Introduction

In this paper we consider *O*-species, which generalize the notion of species introduced by Gabriel in [1]. Recall this definition:

Definition 1.1. (Gabriel [1]). Let I be a finite index set. A **species** $L = (F_i, {}_iM_j)_{i,j \in I}$ is a finite family $(F_i)_{i \in I}$ of division rings together with a family $({}_iM_j)_{i,j \in I}$ of (F_i, F_j)-bimodules.

We say that $(F_i, {}_iM_j)_{i,j \in I}$ is a ***K*-species** if all F_i are finite dimensional and central over the common commutative subfield K which acts centrally on ${}_iM_j$, i.e. $\lambda m = m\lambda$ for all $\lambda \in K$ and all $m \in {}_iM_j$. We also assume that each bimodule ${}_iM_j$ is a finite dimensional vector space over K. K-species is a ***K*-quiver** if $F_i = K$ for each i.

Definition 1.2. A **representation** $(V_i, {}_j\varphi_i)$ of a species $L = (F_i, {}_iM_j)_{i,j \in I}$ (or an ***L*-representation**) is a family of right F_i-modules V_i and F_j-linear mappings:

$$ {}_j\varphi_i : V_i \otimes_{F_i} {}_iM_j \to V_j \tag{1.3}$$

for each $i, j \in I$. Such a representation is called **finite dimensional**, provided all the spaces V_i are finite dimensional vector spaces.

Let $V = (V_i, {}_j\varphi_i)$ and $W = (W_i, {}_j\psi_i)$ be two L-representations. An L-morphism Ψ: $V \to W$ is a set of F_i-linear maps $\alpha_i : V_i \to W_i$ such that

$$ {}_j\psi_i(\alpha_i \otimes 1) = \alpha_j \cdot {}_j\varphi_i \tag{1.4}$$

Two representations $(V_i, {}_j\varphi_i)$ and $W = (W_i, {}_j\psi_i)$ are called **equivalent** if there is a set of isomorphisms α_i from the F_i-module V_i to the F_i-module W_i such that the (1.4) holds for all $i,j \in I$.

A representation $(V_i, {}_j\varphi_i)$ is called **indecomposable**, if there are no non-zero sets of subspaces (U_i) and (W_i) such that $V_i = U_i \oplus W_i$ and ${}_j\varphi_i = {}_j\psi_i \oplus {}_j\tau_i$, where

$$ {}_j\psi_i : U_i \otimes_{F_i} {}_iM_j \to U_j \tag{1.5}$$

$$ {}_j\tau_i : W_i \otimes_{F_i} {}_iM_j \to W_j \tag{1.6}$$

One defines the direct sum of two L-representations in the obvious way.

Denote by Rep(L) the category of all L-representations, and by rep(L) the category of finite dimensional L-representations, whose objects are L-representations and whose morphisms are as defined above.

Definition 1.7. [2] A species $L = (F_i, {}_iM_j)_{i,j\in I}$ is said to be of **finite type**, if the number of indecomposable non-isomorphic finite dimensional representations is finite.

A species $L = (F_i, {}_iM_j)_{i,j\in I}$ is said to be of **strongly unbounded type** if it possesses the following three properties:

1. L has indecomposable objects of arbitrary large finite dimension.
2. If L contains a finite dimensional object with an infinite endomorphism ring, then there is an infinite number of (finite) dimensions d such that, for each d, the species L has infinitely many (non-isomorphic) indecomposable objects of dimension d.
3. L has indecomposable objects of infinite dimension.

Dlab and Ringel proved in [2, Theorem E] that any K-species is either of finite or of strongly unbounded type.

With any species $L = (F_i, {}_iM_j)_{i,j\in I}$ one can define the tensor algebra in the following way. Let $B = \prod_{i\in I} F_i$, and let $M = \bigoplus_{i,j\in I} {}_iM_j$. Then B is a ring and M naturally becomes a (B, B)-bimodule. The **tensor algebra** of the (B, B)-bimodule M is the graded ring

$$T(L) = T_B(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n} \tag{1.8}$$

with component-wise addition and the multiplication induced by taking tensor products.

If L is a K-species, then $T(L)$ is a finite dimensional K-algebra.

Theorem 1.9. (Dlab, Ringel [2, Proposition 10.1]). Let L be a K-species. Then the category Rep(L) of all representations of L and the category $\mathrm{Mod}_r(T(L))$ of all right $T(L)$-modules are equivalent.

2. *O*-species and their representations

In this section we consider the notion of O-species, which generalizes the notion of species considered in [1].

Let $\{O_i\}$ be a family of discrete valuation rings (not necessarily commutative) O_i with radicals R_i and skew fields of fractions D_i, for $i = 1, 2, ..., k$, and let $\{D_j\}$, for $j = k + 1, ..., n$, be a family of skew fields. Let $(n_1, n_2, ..., n_k)$ be a set of natural numbers. Write

$$H_{n_i}(O_i) = \begin{pmatrix} O_i & O_i & \cdots & O_i \\ R_i & O_i & \cdots & O_i \\ \vdots & \vdots & \ddots & \vdots \\ R_i & R_i & \cdots & O_i \end{pmatrix},$$

which is a subring in the matrix ring $M_{n_i}(D_i)$. It is easy to see that each $H_{n_i}(O_i)$ is a Noetherian serial prime hereditary ring. Write $F_i = H_{n_i}(O_i)$ for $i = 1, 2, ..., k$, and $F_j = D_j$ for $j = k + 1, ..., n$. Then, by the Goldie theorem, there exists a classical ring of fractions $\widetilde{F}_i$ for $i = 1,2, ..., n$.

Consider the following generalization of a species.

Definition 2.1. An ***O*-species** is a set $\Omega = (F_i, {}_iM_j)_{i,j \in I}$, where $F_i = H_{n_i}(O_i)$ for $i = 1, 2, ..., k$, and $F_j = D_j$ for $j = k + 1, ..., n$, and moreover ${}_iM_j$ is an $(\widetilde{F}_i, \widetilde{F}_j)$-bimodule, which is finite dimensional as a right D_j-vector space and as a left D_i-vector space.

An O-species Ω is called a **(*D*, *O*)-species** if all O_i have a common skew field of fractions D, i.e. all D_i are equal to a fixed skew field D and

$$_D({}_iM_j)_D \cong ({}_DD_D)^{n_{ij}} \tag{2.2}$$

for some natural number n_{ij} ($i = 1, 2, ..., n$).

An O-species Ω is called a **(K, O)-species**, if all D_i ($i = 1, 2, ..., n$) contain a common central subfield K of finite index in such a way that $\lambda m = m\lambda$ for all $\lambda \in K$ and all $m \in {}_iM_j$ (moreover, each bimodule ${}_iM_j$ is a finite dimensional vector space over K). It is a (K, O)-quiver if moreover $D_i = D$ for each i.

Everywhere in this paper we will consider O-species without oriented cycles and loops, i.e. we will assume that ${}_iM_i = 0$, and if ${}_iM_j \neq 0$, then ${}_jM_i = 0$. A vertex i is said to be **marked** if $F_i = H_{n_i}(O_i)$.

We will also assume that all marked vertices are minimal, i.e. ${}_jM_i = 0$ if $F_i = H_{n_i}(O_i)$, and that ${}_iM_j = {}_jM_i = 0$ if i, j are marked vertices.

Definition 2.3. The **diagram** of an O-species $\Omega = \{F_i, {}_iM_j\}_{i,j\in I}$ is defined in the following way:

1. The set of vertices is a finite set $I = \{1, 2, ..., n\}$.
2. The finite subset $I_0 = \{1, 2, ..., k\}$ of I is a set of marked points.
3. The vertex i connects with the vertex j by t_{ij} arrows, where

$$t_{ij} = \frac{1}{n_i}\dim_D({}_iM_j)\times\dim({}_iM_j)_D + \frac{1}{n_j}\dim_D({}_jM_i)\times\dim({}_jM_i)_D$$

moreover, we assume that $n_i = 1$ if $F_i = D_i$.

Similar to species we can define representations of O-species in the following way.

Definition 2.4. A **representation** $(M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ of an O-species $\Omega = \{F_i, {}_iM_j\}_{i,j\in I}$ is a family of right F_i-modules M_i ($i = 1, 2, ..., k$), a set of right vector spaces V_r over D_r ($r = k+1, k+1, ..., n$) and D_j-linear maps:

$${}_j\varphi_i : M_i \otimes_{F_i} {}_iM_j \to V_j$$

for each $i = 1, 2, ..., k$; $j = k+1, k+2, ..., n$; and

$${}_j\psi_r : V_r \otimes_{D_r} {}_rM_j \to V_j$$

for each $r, j = k+1, k+2, ..., n$.

Definition 2.5. Two representations $M = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ and $M' = (M'_i, V'_r, {}_j\varphi'_i, {}_j\psi'_r)$ are called **equivalent** if there is a set of isomorphisms α_i of F_i-modules from M_i to

M_i' and a set of isomorphisms β_r of D_r-vector spaces from V_r to V_r' such that for each $i = 1, 2, ..., k$; $r, j = k + 1, k + 2, ..., n$ the following equalities hold:

$$ {}_j\varphi_i'(\alpha_i \otimes 1) = \beta_j \cdot {}_j\varphi_i \tag{2.6}$$

$$ {}_j\psi_r'(\beta_r \otimes 1) = \beta_j \cdot {}_j\psi_r \tag{2.7}$$

In a natural way one can define the notions of a direct sum of representations and of an indecomposable representation.

The set of all representations of an O-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ can be turned into a category $R(\Omega)$, whose objects are representations $M = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$, and a morphism from object $M = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ to object $M' = (M_i', V_r', {}_j\varphi_i', {}_j\psi_r')$ is a set of homomorphisms α_i of $H_{n_i}(O_i)$ - modules M_i to M_i', and a set of homomorphisms β_r of D_r - vector spaces from V_r to V_r' such that for each $i = 1,2, ..., k$; $r, j = k + 1, k + 2, ..., n$ the equalities (2.6) and (2.7) hold.

3. Tensor algebra of O-species

For any O-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ one can construct a tensor algebra of bimodules $T(\Omega)$. Let $A = \bigoplus_{i=1}^{n} F_i$, $B = \bigoplus_{i,j} {}_iM_j$. Then B is an (A, A) - bimodule and we can define a tensor algebra $T_A(B)$ of the bimodule B over the ring A in the following way:

$$T_A(B) = A \oplus B \oplus B^2 \oplus ... \oplus B^n \oplus ... \tag{3.1}$$

is a graded ring, where $B^n = B \otimes_A B^{n-1}$ for $n > 1$, and multiplication in $T_A(B)$ is given by the natural A-bilinear map:

$$B^n \times B^m \to B^n \otimes_A B^m = B^{n+m} \tag{3.2}$$

Then $T(\Omega) = T_A(B)$ is the tensor algebra corresponding to an O-species Ω.

Proposition 3.3. Let Ω be an O-species. Then the category $\Re(\Omega)$ of all representations of Ω and the category $\mathrm{Mod_r}\ T(\Omega)$ of all right $T(\Omega)$-modules are naturally equivalent.

Proof. Form two functors R: $\mathrm{Mod_r}\ T(\Omega) \to \Re(\Omega)$ and P: $\Re(\Omega) \to \mathrm{Mod_r}\ T(\Omega)$ in the following way. Let $X_{T(\Omega)}$ be a right $T(\Omega)$-module. Since A is a subring in $T(\Omega)$, X can be considered as a right A-module. Then

$$X = \left(\bigoplus_{i=1}^{k} M_i \right) \oplus \left(\bigoplus_{r=k+1}^{n} V_r \right), \tag{3.4}$$

where M_i is an $H_{n_i}(O_i)$-module, and V_r is a D_r-vector space; moreover, $M_i H_{n_j}(O_j) = 0$ for $i \neq j$, and $V_r D_s = 0$ for $r \neq s$. Since B is an (A, A)-bimodule, one can define an A-homomorphism $\varphi : X \otimes_A B \to X_A$. Taking into account that $M_i \otimes_A {}_sM_j = 0$ for $i \neq s$, the map φ is defined in the following way:

$$\varphi : \left(\bigoplus_{i=1}^{k} (M_i \otimes_{A\,i} M_j) \right) \oplus \left(\bigoplus_{r=k+1}^{n} (V_r \otimes_{A\,r} M_j) \right) \to \bigoplus_{r=k+1}^{n} V_r \tag{3.5}$$

Since $M_i \otimes_A {}_iM_j$ is mapping into V_j, and $V_r \otimes_A {}_rM_j$ is mapping into V_j, φ defines a set of D_j-homomorphisms:

$${}_j\varphi_i : M_i \otimes_A {}_iM_j = M_i \otimes_{H_{n_i}(O_i)} {}_iM_j \to V_j \tag{3.6}$$

$${}_j\psi_r : V_r \otimes_A {}_rM_j = V_r \otimes_{D_r} {}_rM_j \to V_j \tag{3.7}$$

for $i = 1, 2, ..., k$; $r, j = k + 1, ..., n$.

Now one can define $R(X_{T(\Omega)}) = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$. Let X, Y be two right $T(\Omega)$-modules, let $\alpha: X \to Y$ be a homomorphism, and let $R(X) = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$, $R(Y) = (N_i, W_r, {}_j\widetilde{\varphi}_i, {}_j\widetilde{\psi}_r)$. Let's define a morphism from $R(X)$ to $R(Y)$. Since α is an A-homomorphism, $\alpha(M_i) \subseteq N_i$, $\alpha(V_r) \subseteq W_r$, i.e., α defines a family of $H_{n_i}(O_i)$-homomorphisms α_i: $M_i \to N_i$ and a family of D_r-homomorphisms β_r: $V_r \to W_r$, which are the restrictions of α to M_i and V_r. Therefore one can set $R(\alpha) = \{(\alpha_i), (\beta_r)\}$. Since α is a $T(\Omega)$-homomorphism,

$${}_j\widetilde{\varphi}_i(\alpha_i \otimes 1) = \alpha_j \cdot {}_j\varphi_i \tag{3.8}$$

and

$${}_j\widetilde{\psi}_r(\beta_r \otimes 1) = \beta_j \cdot {}_j\psi_r \tag{3.9}$$

for $i = 1, 2, ..., k$; $r, j = k + 1, ..., n$. Therefore $R(\alpha)$ is a morphism in the category $R(\Omega)$.

Conversely, let $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ and there is given a representation $M = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$. Then one can define $P(M)$ in the following way:

$$P(M) = X = \left(\bigoplus_{i=1}^{k} M_i \right) \oplus \left(\bigoplus_{r=k+1}^{n} V_r \right). \tag{3.10}$$

We define an action of

$$A = \left(\bigoplus_{i=1}^{k} H_{n_i}(O_i) \right) \oplus \left(\bigoplus_{r=k+1}^{n} D_r \right) \qquad (3.11)$$

on M_i by means of the projection $A \to H_{n_i}(O_i)$ and an action of A on V_r by means of the projection $A \to D_r$. We define an action of B^n on X by induction of $\varphi^{(n)} : X \otimes_A B^n \to X$ as follows:

$$\varphi^{(1)} = \bigoplus_{i,j} {}_j\varphi_i \bigoplus_{j,r} {}_j\psi_r : X \otimes_A B = \left(\bigoplus_{i=1}^{k} (M_i \otimes_A {}_iM_j) \right) \oplus \left(\bigoplus_{r=k+1}^{n} (V_r \otimes_A {}_rM_j) \right) =$$

$$= \left(\bigoplus_{i=1}^{k} (M_i \otimes_{H_{n_i}(O_i)} {}_iM_j) \right) \oplus \left(\bigoplus_{r=k+1}^{n} (V_r \otimes_{D_r} {}_rM_j) \right) \to \bigoplus_{r=k+1}^{n} V_r \subseteq X.$$

$$\varphi^{(n+1)} = \varphi(\varphi^{(n)} \otimes 1) : X \otimes_A B^{(n+1)} = (X \otimes_A B) \otimes_A B^n \xrightarrow{\varphi^{(n)} \otimes 1} X \otimes_A B \xrightarrow{\varphi} X$$

If $\alpha = \{\{\alpha_i\}, \{\beta_r\}\}$ is a morphism of a representation $M = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ to a representation $M' = (M'_i, V'_r, {}_j\varphi'_i, {}_j\psi'_r)$, $X = P(M)$, $Y = P(M')$, then

$$\varphi = \bigoplus_i \alpha_i \bigoplus_r \beta_r : X = \bigoplus_i M_i \bigoplus_r V_r \to \bigoplus_i M'_i \bigoplus_r V'_r \qquad (3.12)$$

is a $T(\Omega)$-homomorphism and therefore $P(\alpha) = \varphi$.

It is not difficult to show that R, P are mutually inverse functors and they give an equivalence of categories $\mathrm{Mod}_r\, T(\Omega)$ and $\Re(\Omega)$.

Recall that an Artinian ring A is of **finite representation type** if A has only a finite number of indecomposable finitely generated right A-modules up to isomorphism.

A ring A is of (right) **bounded representation type** (see [3, 4]) if there is an upper bound on the number of generators required for indecomposable finitely presented right A-modules.

Denote by $\mu(M_i)$ the minimal number of generators of an $H_{n_i}(O_i)$-module M_i, and denote by $d_r = \dim_{D_r}(V_r)$ the dimension of vector space V_r over D_r. The dimension of a representation $M = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ is the number

$$d = \dim M = \sum_{i=1}^{n} \mu(M_i) + \sum_{r=k+1}^{n} d_r \qquad (3.13)$$

Definition 3.14. An O-species Ω is said to be of **bounded representation type** if the dimensions of its indecomposable finite dimensional representations have an upper bound.

Corollary 3.15. An O-species Ω is of bounded representation type if and only if the tensor algebra $T(\Omega)$ is of bounded representation type.

Proof. If Ω is an O-species of bounded representation type, then there exists $N > 0$ such that $\dim M < N$ for any indecomposable finite dimensional representation M. Then for any finitely generated $T(\Omega)$-module X we have $\mu(X) < N_1$, where N_1 is some fixed number depending on N, i.e. $T(\Omega)$ is a ring of bounded representation type. The converse also holds: if $T(\Omega)$ is a ring of bounded representation type, then Ω is an O-species of bounded representation type.

Corollary 3.16. Let Ω_1 be a D-species, which is a subspecies of a (D, O)-species Ω. If Ω is of bounded representation type, then Ω_1 is of finite type.

Proof. Since Ω is of bounded representation type, each of its subspecies is of bounded representation type as well. So Ω_1 is of bounded representation type, and, by corollary 3.15, its tensor algebra is of bounded representation type, as well. Since Ω_1 is a D-species, its tensor algebra is an Artinian ring. So it is of finite representation type, by [5]. Therefore, Ω_1 is also of finite representation type.

3. Conclusion

In this paper we introduced O-species and the tensor algebras corresponding to them. These O-species are some generalizations of species first introduced by Gabriel in [1]. We consider the notion of a representation of an O-species. In this paper we prove that the category of all representations of O-species Ω and the category of all right modules over a tensor algebra $T(\Omega)$ are naturally equivalent.

References

[1] Gabriel P., Indecomposable representations I, Manuscripta Math. 1972, 6, 71-103.

[2] Dlab V., Ringel C.M., On algebras of finite representation type, Journal of Algebra 1975, 33, 306-394.

[3] Warfield R.B., Serial rings and finitely presented modules, Journal of Algebra 1975, 37, 2, 187-222.

[4] Gubareni N., On right hereditary SPSD-rings of bounded representation type I, Scientific Research of the Institute of Mathematics and Computer Science 2012, 3(11), 57-70.

[5] Auslander M., Representation theory of Artin algebras II, Comm. Algebra 1974, 1, 269-310.