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THE DETERMINANTS OF THE BLOCK BAND MATRICES BASED ON THE *n*-DIMENSIONAL FOURIER EQUATION

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Abstract. This paper contains the method of calculating the determinant of the block band matrix on the example of *n*-dimensional Fourier equation using the FDM.

Keywords: *block matrices, n-band matrices, determinant, Fourier equation*

1. Introduction

In this paper we calculate the determinant of the block matrix in the *n*-dimensional case. We express this determinant by the symmetric polynomials of *m* groups of variables by the symmetric polynomials due to each of these groups.

The Fourier equation describing the heat conduction will serve as an example to illustrate how to calculate the determinants of the block band matrix.

2. Solution of the problem

The *n*-dimensional Fourier equation is the following form

$$\lambda\left(\frac{\partial^2 T(x_1, x_2, ..., x_n, t)}{\partial x_1^2} + \frac{\partial^2 T(x_1, x_2, ..., x_n, t)}{\partial x_2^2} + ... + \frac{\partial^2 T(x_1, x_2, ..., x_n, t)}{\partial x_n^2}\right) = \\ = \rho c \frac{\partial T(x_1, x_2, ..., x_n, t)}{\partial t} \tag{1}$$

where λ is a thermal conductivity, c is a specific heat, ρ is a mass density, T is temperature, $x_1, x_2, ..., x_n$ denote the geometrical co-ordinates and t is time.
Then approximations of the second order partial derivatives using FDM are as follows

$$\begin{aligned}
&\frac{\partial^2 T}{\partial x_1^2} = \frac{T_{i_1-1,i_2,\dots,i_n,l} - 2T_{i_1,i_2,\dots,i_n,l} + T_{i_1+1,i_2,\dots,i_n,l}}{\left(\Delta x_1\right)^2}, \quad 1 \le i_1 \le m_1 - 1 \\
&\frac{\partial^2 T}{\partial x_2^2} = \frac{T_{i_1,i_2-1,\dots,i_n,l} - 2T_{i_1,i_2,\dots,i_n,l} + T_{i_1,i_2+1,\dots,i_n,l}}{\left(\Delta x_2\right)^2}, \quad 1 \le i_2 \le m_2 - 1 \\
&\vdots \\
&\frac{\partial^2 T}{\partial x_n^2} = \frac{T_{i_1,i_2,\dots,i_n-1,l} - 2T_{i_1,i_2,\dots,i_n,l} + T_{i_1,i_2,\dots,i_n+1,l}}{\left(\Delta x_n\right)^2}, \quad 1 \le i_n \le m_n - 1
\end{aligned} \tag{2}$$

and the time derivative approximation takes the following form

$$\frac{\Delta T}{\Delta t} = \frac{T_{i_1,i_2,\dots,i_n,l} - T_{i_1,i_2,\dots,i_n,l-1}}{\Delta t}, \quad 1 \le l \le q \tag{3}$$

Thus, the internal iteration corresponding Fourier equation leads to the following band system of equations

$$\begin{aligned}
&\frac{\lambda}{\left(\Delta x_1\right)^2} T_{i_1-1,i_2,\dots,i_n,l} - \frac{2\lambda}{\left(\Delta x_1\right)^2} T_{i_1,i_2,\dots,i_n,l} + \frac{\lambda}{\left(\Delta x_1\right)^2} T_{i_1+1,i_2,\dots,i_n,l} + \\
&+ \frac{\lambda}{\left(\Delta x_2\right)^2} T_{i_1,i_2-1,\dots,i_n,l} - \frac{2\lambda}{\left(\Delta x_2\right)^2} T_{i_1,i_2,\dots,i_n,l} + \frac{\lambda}{\left(\Delta x_2\right)^2} T_{i_1,i_2+1,\dots,i_n,l} + \dots + \\
&+ \frac{\lambda}{\left(\Delta x_n\right)^2} T_{i_1,i_2,\dots,i_n-1,l} - \frac{2\lambda}{\left(\Delta x_n\right)^2} T_{i_1,i_2,\dots,i_n,l} + \frac{\lambda}{\left(\Delta x_n\right)^2} T_{i_1,i_2,\dots,i_n+1,l} = \\
&= \frac{\rho c}{\Delta t} T_{i_1,i_2,\dots,i_n,l} - \frac{\rho c}{\Delta t} T_{i_1,i_2,\dots,i_n,l-1}
\end{aligned} \tag{4}$$

for each time step l.

The main matrix of this system is a block matrix having the following structure

$$A_n = \begin{bmatrix}
A_{n-1} & I_{n-1} & \dots & \dots & \dots & \dots & \dots \\
I_{n-1} & A_{n-1} & I_{n-1} & \dots & \dots & \dots & \dots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\dots & \dots & \dots & \dots & I_{n-1} & A_{n-1} & I_{n-1} \\
\dots & \dots & \dots & \dots & \dots & I_{n-1} & A_{n-1}
\end{bmatrix}_{m_n \times m_n \text{ (block dimension)}}, \quad n \ge 2 \tag{5}$$

while ($n = 1$)

$$A_1 = \begin{bmatrix} a & 1 & \dots & \dots & \dots & \dots & \dots \\ 1 & a & 1 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & 1 & a & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 & a \end{bmatrix}_{m_1 \times m_1} \tag{6}$$

The elements of the matrix A_1 are in the form

$$a = \frac{2\lambda}{(\Delta x_1)^2} + \frac{2\lambda}{(\Delta x_2)^2} + \ldots + \frac{2\lambda}{(\Delta x_n)^2} + \frac{\rho c}{\Delta t} \tag{7}$$

Then

$$\det A_1 = a^{m_1} - \binom{m_1 - 1}{1} a^{m_1 - 2} + \binom{m_1 - 2}{2} a^{m_1 - 4} + \ldots \tag{8}$$

The calculation of the above determinant is given in the article [1]. Then

$$A_2 = \begin{bmatrix} A_1 & I_1 & \dots & \dots & \dots & \dots & \dots \\ I_1 & A_1 & I_1 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & I_1 & A_1 & I_1 \\ \dots & \dots & \dots & \dots & \dots & I_1 & A_1 \end{bmatrix}_{m_2 \times m_2} \tag{9}$$

where

$$I_1 = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 1 \end{bmatrix}_{m_1 \times m_1} \tag{10}$$

Therefore,

$$\begin{aligned} \det A_2 &= \det\left[A_1^{m_2} - \binom{m_2 - 1}{1} A_1^{m_2 - 2} + \binom{m_2 - 2}{2} A_1^{m_2 - 4} - \ldots \right] = \\ &= \det\left[(A_1 - p_{1,1} I_1)(A_1 - p_{1,2} I_1) \ldots (A_1 - p_{1,m_2} I_1) \right] = \\ &= \det(A_1 - p_{1,1} I_1) \cdot \det(A_1 - p_{1,2} I_1) \cdot \ldots \cdot \det(A_1 - p_{1,m_2} I_1) \end{aligned} \tag{11}$$

where p_{1,i_1}, $1 \le i_1 \le m_2$, are zeros of polynomial

$$f_1(x) = x^{m_2} - \binom{m_2 - 1}{1} x^{m_2 - 2} + \binom{m_2 - 2}{2} x^{m_2 - 4} + \ldots = \\ = \left(x - p_{1,1}\right) \cdot \left(x - p_{1,2}\right) \cdot \ldots \cdot \left(x - p_{1,m_2}\right) \tag{12}$$

Consecutively,

$$A_3 = \begin{bmatrix} A_2 & I_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_2 & A_2 & I_2 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & I_2 & A_2 & I_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & I_2 & A_2 \end{bmatrix}_{m_3 \times m_3} \tag{13}$$

where

$$I_2 = \begin{bmatrix} I_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & I_1 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 0 & I_1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & I_1 \end{bmatrix}_{m_2 \times m_2} \tag{14}$$

Then we obtain [2]

$$\det A_3 = \det\left[A_2^{m_3} - \binom{m_3 - 1}{1} A_2^{m_3 - 2} + \binom{m_3 - 2}{2} A_2^{m_3 - 4} - \ldots \right] = \\ = \det\left[\left(A_2 - p_{2,1} I_2\right)\left(A_2 - p_{2,2} I_2\right) \ldots \left(A_2 - p_{2,m_3} I_2\right) \right] = \\ = \det\left(A_2 - p_{2,1} I_2\right) \cdot \det\left(A_2 - p_{2,2} I_2\right) \cdot \ldots \cdot \det\left(A_2 - p_{2,m_3} I_2\right) \tag{15}$$

where p_{2,i_2}, $1 \le i_2 \le m_3$, are zeros of polynomial

$$f_2(x) = x^{m_3} - \binom{m_3 - 1}{1} x^{m_3 - 2} + \binom{m_3 - 2}{2} x^{m_3 - 4} + \ldots = \\ = \left(x - p_{2,1}\right) \cdot \left(x - p_{2,2}\right) \cdot \ldots \cdot \left(x - p_{2,m_3}\right) \tag{16}$$

Generally,

$$\det A_n = \det\left[A_{n-1}{}^{m_n} - \binom{m_n - 1}{1} A_{n-1}{}^{m_n-2} + \binom{m_n - 2}{2} A_{n-1}{}^{m_n-4} - \ldots \right] =$$
$$= \det\left[\left(A_{n-1} - p_{n-1,1} I_{n-1}\right)\left(A_{n-1} - p_{n-1,2} I_{n-1}\right)\cdot \ldots \cdot \left(A_{n-1} - p_{n-1,m_n} I_{n-1}\right)\right] = \tag{17}$$
$$= \det\left(A_{n-1} - p_{n-1,1} I_{n-1}\right)\cdot \det\left(A_{n-1} - p_{n-1,2} I_{n-1}\right)\cdot \ldots \cdot \det\left(A_{n-1} - p_{n-1,m_n} I_{n-1}\right)$$

where $p_{n-1,i_{n-1}}$, $1 \le i_{n-1} \le m_n$, are zeros of polynomial

$$f_{n-1}(x) = x^{m_n} - \binom{m_n - 1}{1} x^{m_n-2} + \binom{m_n - 2}{2} x^{m_n-4} + \ldots =$$
$$= \left(x - p_{n-1,1}\right)\cdot\left(x - p_{n-1,2}\right)\cdot \ldots \cdot \left(x - p_{n-1,m_n}\right) \tag{18}$$

The first element of the product (17) is as follows

$$\det\left(A_{n-1} - p_{n-1,1} I_{n-1}\right) =$$
$$= \det\left[\left(A_{n-2} - p_{n-2,1} I_{n-2}\right) - p_{n-1,1} I_{n-2} \right]\cdot$$
$$\cdot \det\left[\left(A_{n-2} - p_{n-2,2} I_{n-2}\right) - p_{n-1,1} I_{n-2} \right]\cdot \ldots\ldots\ldots \cdot$$
$$\cdot \det\left[\left(A_{n-2} - p_{n-2,m_{n-2}} I_{n-2}\right) - p_{n-1,1} I_{n-2} \right] = \tag{19}$$
$$= \det\left[A_{n-2} - \left(p_{n-2,1} + p_{n-1,1}\right) I_{n-2} \right]\cdot$$
$$\cdot \det\left[A_{n-2} - \left(p_{n-2,2} + p_{n-1,1}\right) I_{n-2} \right]\cdot \ldots\ldots\ldots \cdot$$
$$\cdot \det\left[A_{n-2} - \left(p_{n-2,m_{n-2}} + p_{n-1,1}\right) I_{n-2} \right]$$

The above product includes m_{n-2} factors, where the first of them is equal to

$$\det\left[A_{n-2} - \left(p_{n-2,1} + p_{n-1,1}\right) I_{n-2} \right] =$$
$$= \det\left[A_{n-3} - \left(p_{n-3,1} + p_{n-2,1} + p_{n-1,1}\right) I_{n-3} \right]\cdot$$
$$\cdot \det\left[A_{n-3} - \left(p_{n-3,2} + p_{n-2,1} + p_{n-1,1}\right) I_{n-3} \right]\cdot \ldots\ldots\ldots \cdot \tag{20}$$
$$\cdot \det\left[A_{n-3} - \left(p_{n-3,m_{n-3}} + p_{n-2,1} + p_{n-1,1}\right) I_{n-3} \right]$$

The resulting product contains m_{n-3} factors and the first of these factors is as follows

$$\begin{aligned}
&\det\left[A_{n-3}-\left(p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)I_{n-3}\right]= \\
&=\det\left[A_{n-4}-\left(p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)I_{n-4}\right]\cdot \\
&\cdot\det\left[A_{n-4}-\left(p_{n-4,2}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)I_{n-4}\right]\cdot\ldots\ldots\ldots\cdot \\
&\cdot\det\left[A_{n-4}-\left(p_{n-4,m_{n-4}}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)I_{n-4}\right]
\end{aligned} \tag{21}$$

where we have m_{n-4} factors.

Consequently, the initial factor in (19) takes a form

$$\begin{aligned}
&\det\left[A_1-\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)I_1\right]\cdot \\
&\cdot\det\left[A_1-\left(p_{1,2}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)I_1\right]\cdot\ldots\ldots\ldots\cdot \\
&\cdot\det\left[A_1-\left(p_{1,m_1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)I_1\right]= \\
&=W_{A_1}\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)\cdot \\
&\cdot W_{A_1}\left(p_{1,2}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)\cdot\ldots\ldots\ldots\cdot \\
&\cdot W_{A_1}\left(p_{1,m_1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)
\end{aligned} \tag{22}$$

wherein the first factor of m_1 factors of the above formula takes the form

$$\begin{aligned}
&W_{A_1}\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)= \\
&=(-1)^{m_1}\left[\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)-\lambda_1\right]\cdot \\
&\cdot\left[\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)-\lambda_2\right]\cdot\ldots\ldots\ldots\cdot \\
&\cdot\left[\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}\right)-\lambda_{m_1}\right]= \\
&=\lambda_1\cdot\lambda_2\cdot\ldots\cdot\lambda_{m_1}\left(1-\frac{p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}}{\lambda_1}\right)\cdot \\
&\qquad\cdot\left(1-\frac{p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}}{\lambda_2}\right)\cdot\ldots\ldots\ldots\cdot \\
&\qquad\cdot\left(1-\frac{p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,1}}{\lambda_{m_1}}\right)
\end{aligned} \tag{23}$$

According to the above procedure we present the second factor of the formula (17)

$$\begin{aligned}
&\det\left(A_{n-1}-p_{n-1,2}I_{n-1}\right)=\\
&=\det\left[\left(A_{n-2}-p_{n-2,1}I_{n-2}\right)-p_{n-1,2}I_{n-2}\right]\cdot\\
&\cdot\det\left[\left(A_{n-2}-p_{n-2,2}I_{n-2}\right)-p_{n-1,2}I_{n-2}\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[\left(A_{n-2}-p_{n-2,m_{n-2}}I_{n-2}\right)-p_{n-1,2}I_{n-2}\right]=\\
&=\det\left[A_{n-2}-\left(p_{n-2,1}+p_{n-1,2}\right)I_{n-2}\right]\cdot\\
&\cdot\det\left[A_{n-2}-\left(p_{n-2,2}+p_{n-1,2}\right)I_{n-2}\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[A_{n-2}-\left(p_{n-2,m_{n-2}}+p_{n-1,2}\right)I_{n-2}\right]
\end{aligned}\tag{24}$$

where the first factor of above product is equal to

$$\begin{aligned}
&\det\left[A_{n-2}-\left(p_{n-2,1}+p_{n-1,2}\right)I_{n-2}\right]=\\
&=\det\left[A_{n-3}-\left(p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_{n-3}\right]\cdot\\
&\cdot\det\left[A_{n-3}-\left(p_{n-3,2}+p_{n-2,1}+p_{n-1,2}\right)I_{n-3}\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[A_{n-3}-\left(p_{n-3,m_{n-3}}+p_{n-2,1}+p_{n-1,2}\right)I_{n-3}\right]
\end{aligned}\tag{25}$$

and once again we have the first factor of the product (25)

$$\begin{aligned}
&\det\left[A_{n-3}-\left(p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_{n-3}\right]=\\
&=\det\left[A_{n-4}-\left(p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_{n-4}\right]\cdot\\
&\cdot\det\left[A_{n-4}-\left(p_{n-4,2}+p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_{n-4}\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[A_{n-4}-\left(p_{n-4,m_{n-4}}+p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_{n-4}\right]
\end{aligned}\tag{26}$$

The initial factor in (24) takes a form

$$\begin{aligned}
&\det\left[A_1-\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_1\right]\cdot\\
&\cdot\det\left[A_1-\left(p_{1,2}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_1\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[A_1-\left(p_{1,m_1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,2}\right)I_1\right]=
\end{aligned}\tag{27}$$

$$= W_{A_1}\left(p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}\right)\cdot$$
$$\cdot W_{A_1}\left(p_{1,2} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}\right)\cdot \ldots\ldots\ldots \cdot$$
$$\cdot W_{A_1}\left(p_{1,m_1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}\right)$$

For example, we calculate the first factor

$$W_{A_1}\left(p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}\right) =$$
$$= (-1)^{m_1}\left[\left(p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}\right) - \lambda_1\right]\cdot$$
$$\cdot\left[\left(p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}\right) - \lambda_2\right]\cdot \ldots\ldots\ldots \cdot$$
$$\cdot\left[\left(p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}\right) - \lambda_{m_1}\right] =$$
$$= \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_{m_1}\left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}}{\lambda_1}\right)\cdot$$
$$\cdot\left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}}{\lambda_2}\right)\cdot \ldots\ldots\ldots \cdot$$
$$\cdot\left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,2}}{\lambda_{m_1}}\right) \tag{28}$$

and the last element of the formula (17) is in the form

$$\det\left(A_{n-1} - p_{n-1,m_n} I_{n-1}\right) =$$
$$= \det\left[\left(A_{n-2} - p_{n-2,1} I_{n-2}\right) - p_{n-1,m_n} I_{n-2}\right]\cdot$$
$$\cdot\det\left[\left(A_{n-2} - p_{n-2,2} I_{n-2}\right) - p_{n-1,m_n} I_{n-2}\right]\cdot \ldots\ldots\ldots \cdot$$
$$\cdot\det\left[\left(A_{n-2} - p_{n-2,m_{n-2}} I_{n-2}\right) - p_{n-1,m_n} I_{n-2}\right] = \tag{29}$$
$$= \det\left[A_{n-2} - \left(p_{n-2,1} + p_{n-1,m_n}\right) I_{n-2}\right]\cdot$$
$$\cdot\det\left[A_{n-2} - \left(p_{n-2,2} + p_{n-1,m_n}\right) I_{n-2}\right]\cdot \ldots\ldots\ldots \cdot$$
$$\cdot\det\left[A_{n-2} - \left(p_{n-2,m_{n-2}} + p_{n-1,m_n}\right) I_{n-2}\right]$$

where

$$\begin{aligned}
&\det\left[A_{n-2}-\left(p_{n-2,1}+p_{n-1,m_n}\right)I_{n-2}\right]=\\
&=\det\left[A_{n-3}-\left(p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)I_{n-3}\right]\cdot\\
&\cdot\det\left[A_{n-3}-\left(p_{n-3,2}+p_{n-2,1}+p_{n-1,m_n}\right)I_{n-3}\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[A_{n-3}-\left(p_{n-3,m_{n-3}}+p_{n-2,1}+p_{n-1,m_n}\right)I_{n-3}\right]
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
&\det\left[A_{n-3}-\left(p_{n-3,1}+p_{n-2,1}+p_{n-1,m_{n-1}}\right)I_{n-3}\right]=\\
&=\det\left[A_{n-4}-\left(p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,}\right)I_{n-4}\right]\cdot\\
&\cdot\det\left[A_{n-4}-\left(p_{n-4,2}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_{n-1}}\right)I_{n-4}\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[A_{n-4}-\left(p_{n-4,m_{n-4}}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_{n-1}}\right)I_{n-4}\right]
\end{aligned}\tag{31}$$

Finally, we obtain this formula

$$\begin{aligned}
&\det\left[A_1-\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)I_1\right]\cdot\\
&\cdot\det\left[A_1-\left(p_{1,2}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)I_1\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\det\left[A_1-\left(p_{1,m_1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_{n-}}\right)I_1\right]=\\
&=W_{A_1}\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)\cdot\\
&\cdot W_{A_1}\left(p_{1,2}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)\cdot\ldots\ldots\ldots\cdot\\
&\cdot W_{A_1}\left(p_{1,m_1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)
\end{aligned}\tag{32}$$

where

$$\begin{aligned}
&W_{A_1}\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)=\\
&=(-1)^{m_1}\left[\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)-\lambda_1\right]\cdot\\
&\cdot\left[\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)-\lambda_2\right]\cdot\ldots\ldots\ldots\cdot\\
&\cdot\left[\left(p_{1,1}+p_{2,1}+\ldots+p_{n-4,1}+p_{n-3,1}+p_{n-2,1}+p_{n-1,m_n}\right)-\lambda_{m_1}\right]=
\end{aligned}\tag{33}$$

$$= \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_{m_1} \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,m_n}}{\lambda_1} \right) \cdot$$

$$\cdot \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,m_n}}{\lambda_2} \right) \cdot \ldots \ldots \ldots \cdot$$

$$\cdot \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,m_n}}{\lambda_{m_1}} \right)$$

Therefore, we received the symmetric polynomial due to *m* groups of variables $\lambda_1, \lambda_2, \ldots, \lambda_{m_1}; p_{1,1}, p_{1,2}, \ldots, p_{1,m_2}; \ldots; p_{m-1,1}, p_{m-1,2}, \ldots, p_{m-1,m_n}$.
While

$$\det A_n = \left(\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_{m_1} \right)^{m_1 \cdot m_2 \cdot \ldots \cdot m_n} \cdot W_n \tag{34}$$

where

$$W_n = \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,1}}{\lambda_1} \right) \cdot$$

$$\cdot \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,1}}{\lambda_2} \right) \cdot \ldots \ldots \ldots \cdot$$

$$\cdot \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,1}}{\lambda_{m_1}} \right) \cdot \ldots \ldots \ldots \cdot$$

$$\cdot \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,m_n}}{\lambda_1} \right) \cdot \tag{35}$$

$$\cdot \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,m_n}}{\lambda_2} \right) \cdot \ldots \ldots \ldots \cdot$$

$$\cdot \left(1 - \frac{p_{1,1} + p_{2,1} + \ldots + p_{n-4,1} + p_{n-3,1} + p_{n-2,1} + p_{n-1,m_n}}{\lambda_{m_1}} \right)$$

$$= 1 - S_1 + S_2 - \ldots + (-1)^{m_1 \cdot m_2 \cdot \ldots \cdot m_n} S_{m_1 \cdot m_2 \cdot \ldots \cdot m_n}$$

where S_1 (the first symmetric polynomial of the indicated above variables) is equal to

$$S_1 = \sum_{\substack{1\le j\le m_1\\ 1\le i_1\le m_2\\ 1\le i_2\le m_3\\ \dots\dots\dots\\ 1\le i_{n-1}\le m_n}} \frac{p_{i_1,1}+p_{i_2,2}+\dots+p_{i_{n-1},m}}{\lambda_j} = m_2\cdot\ldots\cdot m_n\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\dots+\frac{1}{\lambda_{m_1}}\right)\left(\underbrace{p_{1,1}+p_{2,1}+\dots+p_{m_2,1}}_{\omega_{1,1}}\right)\cdot\ldots\cdot$$

$$\cdot\left(\underbrace{p_{1,m}+p_{2,m}+\dots+p_{m_n,m}}_{\omega_{1,n-1}}\right) = m_2\cdot\ldots\cdot m_n\frac{\tau_{m_1-1}}{\tau_{m_1}}\cdot\omega_{1,1}\cdot\ldots\cdot\omega_{1,n-1} \tag{36}$$

where

$\tau_{m_1-1} = \tau_{m_1-1}\left(\lambda_1, \lambda_2, \dots, \lambda_{m_1}\right)$, $\tau_{m_1} = \tau_{m_1}\left(\lambda_1, \lambda_2, \dots, \lambda_{m_1}\right)$ - the fundamental symmetric polynomials of the rank m_1-1, m_1 respectively,

$\omega_{1,1}, \dots, \omega_{1,n-1}$ - the first rank fundamental symmetric polynomials with respect to the groups of variables $p_{1,1}, p_{1,2}, \dots, p_{1,m_2}; \dots; p_{m-1,1}, p_{m-1,2}, \dots, p_{m-1,m_n}$.

Other symmetric polynomials we count based on the repeated application of Newton's formulas [3]. Examples of calculations we have given in articles describing 2D and 3D cases [4, 5].

However, the general schemes require rather laborious and time-consuming computational techniques.

3. Conclusion

The article describes the procedure for calculating the determinant of the main block matrix occurring in the system of equations for internal nodes of the area described by the *n*-dimensional Fourier equation.

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