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APPLICATIONS OF SENSITIVITY ANALYSIS IN MULTI-OBJECTIVE LINEAR PROGRAMMING PROBLEMS

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Abstract

The paper presents multi-objective linear programming problems and analyzes the sensitivity of their efficient solutions. The aim is to apply sensitivity analysis methods to real-life problems. The novelty of this work is based on the integration of results from the following fields: sensitivity analysis theory, multi-objective optimization and real-life problems. The contribution is the development of potential applications for three approaches: the tolerance approach, the standard approach and the range set approach. The proposed methodology is demonstrated using four examples: the transportation problem, the project management problem, the manufacturing problem and the diet problem.

Keywords: multi-objective linear programming (MOLP), sensitivity analysis (SA), transportation problem, project management, diet problem, manufacturing problem.

1 Introduction

The paper considers multi-objective linear programming (MOLP) problems, which have numerous practical applications. Several examples of MOLP applications can be found in the literature. Mardani et al. (2017) describe energy management problems over two decades, from 1995 to 2015. A literature review in the field of environmental sciences is provided by Cegan et al. (2017). Sriram et al. (2022) discuss applications in high-tech market sectors. Sahoo and Goswami (2023) present a comprehensive review of MOLP in the context of multi-criteria decision-making, outlining various applications and future research directions. In all these applications, identifying efficient solutions is essential. Consequently, analyzing these solutions in detail becomes important. In this

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paper, we address this issue by focusing on sensitivity analysis (SA). Applications of SA to real-life problems are presented in several papers: Berna-Escriche et al. (2023) apply SA to energy-related problems; Saidi et al. (2023) investigate its use in biodiesel production and Emara, Armanuos, and Shalby (2024) describe its role in hydro-chemical systems.

SA is also a significant component of linear programming. Numerous papers describe SA in single-criterion linear programming problems; for example, see the historical overview provided by Gal (1997). Furthermore, SA can be formulated in various ways including:

- interval parameters: Rivaz and Saeidi (2021), Van Tu (2024),
- fuzzy parameters: Perić, Babić and Hunjak (2023), Supian et al. (2024),
- stochastic approach: Lam (2016), Arabjazi et al. (2021).

Moreover, SA is an important support tool in decision-making for MOLP problems. In such problems, the focus is on efficient (Pareto optimal) solutions. MOLP problems can have many efficient solutions. Therefore, making a final decision involves considering multiple factors, one of which is SA. A more stable (less sensitive) efficient solution may be preferable for the decision maker. Furthermore, in the context of changing parameters in MOLP problems, SA makes it also possible to study the evolution of the entire set of efficient solutions. In real-life problems, the coefficients of inputs (such as costs, product prices, and other parameters) may be estimated. In such cases, SA describes how changes in the coefficients can cause a given solution to lose its efficiency. SA can also support the decision maker in interactive methods, where not only preferences may change, but also parameters that arise during problem analysis.

In this paper, we present a problem in which an efficient solution was chosen, with a focus on SA of this solution. Moreover, we consider SA with real-valued parameters that characterize MOLP problems. This approach addresses the sensitivity of efficient solutions in MOLP, specifically focusing on changes in the coefficients of the objective functions. Our research aims to demonstrate the applicability of this approach to real-life problems. To achieve this, we consider three approaches of SA for MOLP problems:

- tolerance approach: Benson (1985), Hladik (2008),
- standard approach: Sitarz (2011; 2014),
- range set approach: Hladik et al. (2019).

Using the above approaches, we consider four examples: the transportation problem, the project management problem, the manufacturing problem, and the diet problem. Additionally, an analysis of the results from the four applications is presented. This analysis provides a deeper understanding of the approaches. Beyond the examples discussed, these approaches can be applied to any MOLP problem where parameter changes are important.

2 SA in MOLP

2.1 Description of the MOLP

We consider the MOLP problem defined as follows (Steuer, 1986):

$$VMax\,\{Cx\colon x \in X\} \tag{1}$$

Where the set $X = \{x \in \mathbb{R}^n\colon Ax \leq b, x \geq 0\} \subset \mathbb{R}^n$ represents the feasible solutions, $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$, and matrix $C \in \mathbb{R}^{n \times k}$ represents the linear objective functions $c^i x$ for $i = 1, \dots, n$.

A feasible solution $x^* \in X$ is called an efficient solution if there does not exist a feasible solution $x \in X$ such that:

$$Cx^* \leq Cx \,\wedge\, Cx^* \neq Cx$$

We focus on SA of an efficient solution x^*under changes in matrix C. In other words, we analyse the impact of changes in the objective functions coefficients on the efficiency of the given solution x^*.

2.2 Percentage and additive tolerance approach

We consider the approach to SA based on the concept of tolerance. This approach, originally introduced in linear programming by Wendell (1982), has been extended to MOLP by Hladik (2008) and Sitarz (2010). It aims to calculate a value (a tolerance) representing the perturbation that can be simultaneously applied to matrix C without affecting the efficiency of a given x^*. We consider two types of tolerances:

- additive: representing a perturbation in the form of additive changes,
- percentage: representing a perturbation in the form of percentage changes.

To define the tolerances formally, we introduce the δ, H-neighbourhood of matrix $C = [c_{ij}]$:

$$O_{\delta,H}(C) = \left\{D = \left[d_{ij}\right] \in \mathbb{R}^{n \times k}\colon \left|d_{ij} - c_{ij}\right| < \delta\left|h_{ij}\right| \text{if} h_{ij} \neq 0,\;\; d_{ij} = c_{ij} \text{if} h_{ij} = 0\right\}$$

In this case we focus on the following problem, obtained from (1) by applying matrix $D \in O_{\delta,H}(C)$:

$$VMax\,\{Dx\colon x \in X\} \tag{2}$$

Definition 1. A tolerance for an efficient solution $x^*$is any real δ such that x^* remains efficient for problem (2) for all $D \in O_{\delta,H}(C)$. The maximal tolerance is denoted by δ_{max}.

We consider the following two types of tolerance: additive tolerance and percentage tolerance, defined as follows:

Definition 2. An additive tolerance is a tolerance defined for the matrix H of the form:

$h_{ij} = 1$ – if coefficient c_{ij} can be changed

$h_{ij} = 0$ – if coefficient c_{ij} does not change

Definition 3. A percentage tolerance is a tolerance defined for the matrix H of the form:

$h_{ij} = c_{ij}$ – if coefficient c_{ij} can be changed

$h_{ij} = 0$ – if coefficient c_{ij} does not change

Remark 1. A detailed description of the tolerance approach, along with some computational methods, is formulated in papers by Hladik (2008), Hladik and Sitarz (2010). Moreover, a software package for computing maximal tolerances (both percentage and additive) is presented by Sitarz and Botor (2021). Paratane and Bit (2020) consider a symmetric tolerance approach applied to the transportation problem.

2.3 Standard approach

We consider the standard approach to SA as an extension of the standard linear programming approach (Gal, 1995). The earliest applications of this approach to MOLP can be found in the papers by Sitarz (2008; 2010). The standard approach aims to find values of a selected objective function coefficient that can be changed without affecting efficiency of a given x^*. In this case, we consider the following problem, obtained by reformulating problem (1):

$$VMax\,\{D_t^{qp}x \colon x \in X\} \tag{3}$$

where matrix $D_t^{qp} = \left[d_{ij}\right] \in \mathbb{R}^{n\,x\,k}$ is obtained from matrix $C = [c_{ij}]$ by changing only one coefficient to the value d:

$$d_{ij} = \begin{cases} c_{ij}\text{if } \ (i,j) \neq (q,p) \\ d \qquad \text{if}(i,j) = (q,p) \end{cases}$$

Definition 4. The set of values d for which a given $x^* \in$ X remains efficient for (3) is defined as follows:

$$T_{x^*} = \{d \in \mathbb{R} \colon x^* \text{ is an efficient solution to (3)}\}$$

Remark 2. The set T_{x^*} is an interval (Sitarz, 2008). Moreover, a detailed description of the standard SA in MOLP, along with some computational methods, is presented in the paper by Sitarz (2008; 2010). The standard SA related to weak efficiency is described by Sitarz (2011). Furthermore, Kaci and Radjef (2022) and Kaci (2024) presents the standard SA from a geometric perspective.

2.4 Range set approach

The range set approach applied to SA has its origins in linear programming theory concerning optimal solution (Gass, 1975; Gal, 1995). The use of this approach in MOLP was proposed by Benson (1985). The range set approach aims to find a parameter set which can be applied to matrix C without affecting the efficiency of a given x^*. In this case we consider the following parameterization of problem (1):

$$VMax\,\{(C+tG)x:\ x\in X\} \tag{4}$$

where $t \in \mathbb{R}$ is a parameter and $G \in \mathbb{R}^{n\,x\,k}$ is a given matrix.

Definition 5. We define the range set as the set of parameters t for which a given $x^* \in$ X is efficient for problem (4):

$$TH(x^*) = \{t \in \mathbb{R}: :\ x^* \text{ is an efficient solution to } (4)\}$$

Remark 3. The paper by Hladik et al. (2019) presents methods for computing the range set using the parametric polytope intersection problem. Some properties of the range sets are given by Sitarz (2014). The range set approach can be formulated as an interval MOLP; this approach is presented by Henriques et al. (2019) and Batamiz, Allahdadi and Hladík (2020).

2.5 Strengths and limitations of the presented approaches

Table 1 presents the strengths and limitations of the proposed approaches. The information contained in this table can help decision makers in selecting the most suitable approach.

Table 1: Strengths and limitations of the presented approaches

Approaches	Strengths	Limitations
The tolerance approach	• The final result is presented as a single number: the feasible perturbation of the given coefficients • Any number of coefficients of the objective functions can be taken into account simultaneously • A choice can be made between percentage and additive perturbation	• Different units of the objective functions influence the final result • It cannot be used dynamically
The standard approach	• The final result is presented as a single interval: the feasible perturbation of the given coefficient • The units of the objective functions do not affect the final result • Each coefficient can be treated separately	• Only one coefficient of the objective functions is taken into account • It cannot be used dynamically
The range set approach	• The final result is presented as a single interval: the feasible perturbation of the parameter t • Dynamic application is possible using the parameter t	• Precise parameter values are required for each coefficient of the objective function • It does not provide a deeper analysis of the objective function coefficients; it only focuses on the parameter t

3 Application in project management problems

We consider project management problems through time-cost analysis. General models of project management are presented in the work by Knutson and Bitz (1991). Time-cost analysis is a generalization of the Critical Path Method (CPM), a well-known approach to project management; the CPM approach is described, for example, in the work by Woolf (2007). Let us consider a project from the paper by Elmabrouk (2011). Table 2 lists various activities required to replace an existing boiler with an energy-efficient one. It also presents the remaining data used in the time-cost analysis for project management. We take into account two criteria:

- minimizing the project completion time,
- minimizing the crash cost.

Table 2: Data for the considered project management

Activity with description	**Depends on**	**Normal time (days)**	**Crash cost per day**	**Maximal crashed days**
A – Preparation of technical specifications	–	10	550	2
B – Tender processing	A	25	3000	1
C – Release of work orders	B	3	1300	1
D – Supply of boiler equipment	C	60	700	2
E – Supply of auxiliaries	C	20	1000	1
F – Supply of pipes & pipe fittings	C	10	220	1
G – Civil work	C	15	1000	2
H – Installation of auxiliary equipment & piping	E, F, G	5	1000	1
I – Installation of boiler	D, H	10	1000	1
J – Testing and commissioning	I	2	1000	1

The MOLP problem constructed for the project from Table 2 is as follows:

$$\text{Min (cost)} 550x_A + 3000x_B + 1300x_C + 700x_D + 1000x_E + 2200x_F + \\ + 1000x_G + 1000x_H + 1000x_I + 1000x_J$$

$$\text{Min (time) } y_{finish}$$

$$y_B + x_A \geq 10, \quad y_C - y_B + x_B \geq 25, \quad y_D - y_C + x_C \geq 3,$$
$$y_E - y_C + x_C \geq 3, \quad y_F - y_C + x_C \geq 3, \quad y_G - y_C + x_C \geq 3,$$
$$y_H - y_G + x_G \geq 15, \quad y_H - y_F + x_F \geq 10, \quad y_H - y_E + x_E \geq 20,$$
$$y_I - y_H + x_H \geq 5, \quad y_I - y_D + x_D \geq 60, \quad y_J - y_I + x_I \geq 10,$$
$$y_{finish} - y_J + x_J \geq 2,$$
$$0 \leq x_A \leq 2, \quad 0 \leq x_B \leq 1, \quad 0 \leq x_C \leq 1, \quad 0 \leq x_D \leq 2,$$
$$0 \leq x_E \leq 1,$$

$$0 \le x_F \le 1, \quad 0 \le x_G \le 2, \quad 0 \le x_H \le 1, \quad 0 \le x_I \le 1,$$
$$0 \le x_J \le 1$$
$$0 \le y_B, \quad 0 \le y_C, \quad 0 \le y_D, \quad 0 \le y_E, \quad 0 \le y_F,$$
$$0 \le y_G, \quad 0 \le y_H, \quad 0 \le y_I, \quad 0 \le y_J$$

We analyse the sensitivity of the following efficient solution $[x^*, y^*]$:

$$x_B^* = x_C^* = x_E^* = x_F^* = x_G^* = x_H^* = 0,$$
$$x_A^* = 2, x_D^* = 2, x_I^* = 1, x_J^* = 1,$$
$$y_B^* = 8, y_C^* = 33, y_D^* = 36, y_E^* = 69, y_F^* = 79, y_G^* = 36, y_H^* = 89, y_I^* = 94,$$
$$y_J^* = 103,$$
$$y_{finish}^* = 104$$

It is worth mentioning that the values of objective functions for this solution are: 4500 (cost) and 104 (time).

We present the results of SA using the additive tolerance approach. It should be noted that in real-life problems the coefficients of the first objective function (costs) may change. Hence, only these coefficients are permitted to change. To construct matrix H, used in the additive tolerance approach, we use the following values for the coefficients of H:

$$h_{ij} = 1 \text{ – for coefficients of the first objective function (cost)}$$
$$h_{ij} = 0 \text{ – for the remaining coefficients}$$

In this case, the maximal additive tolerance for the solution $[x^*, y^*]$ is $\delta_{max} =$ = 1.067. This means that $[x^*, y^*]$ remains efficient when cost coefficients vary simultaneously within ±1.067. To illustrate these results, we calculate the interval for the coefficient $c_{1,3} = 1300$ as follows:

$$[c_{1,3} - \delta_{max},\ c_{1,3} + \delta_{max}] = [1300 - 1.067,\ 1300 + 1.067] = [1298.933,\ 1301.067]$$

Analogously, we calculate intervals for the other coefficients.

Remark 4. The obtained intervals show the possible perturbations of the coefficients for which the considered solution $[\mathrm{x}^*, \mathrm{y}^*]$ remains efficient (in the additive sense). Notice that a single number $\delta_{\max}$ represents the simultaneous perturbations of all coefficients. Moreover, if many efficient solutions are considered, $\delta_{\max}$ can be calculated for each one. A higher value of $\delta_{\max}$ means a more stable (less sensitive) efficient solution. This information is relevant to decision makers who take sensitivity into account. In such cases, the decision maker prefers the efficient solution with the maximal value of $\delta_{\max}$.

4 Application in the diet problem

We consider the well-known linear programming problem called the diet problem. This problem is analyzed as a single-criterion problem in many papers, for example by Gass (1970; 1975). In the case of bi-criteria linear programming, the diet problem is studied by Benson and Morin (1987) in the context of nutritional planning in developing countries. The MOLP approach to the sausage production problem is presented by Steuer (1984).

Here, we focus on the "Optimal Nutrition" model proposed by Kwasniewski (1999). There are strict rules about the proportions of the three main food components: protein (P), fat (F), and carbohydrates (C). The ideal ratio between these components is:

1(P): 2.5-3.5(F): 0.8 (C)

Also, the recommended amount of protein per day is 1 gram for each kilogram of body weight (BW). In our model, BW is a constant and is written as $const_{\mathrm{BW}}$. Based on these rules, we take into account two criteria:

- minimizing diet cost,
- maximizing food energy.

To formulate the MOLP problem, we need to use food products with known amounts of protein, fat, carbohydrates, and food energy. To illustrate this numerically, we consider a set of ten basic food products:

- butter (x_1),
- bacon (x_2),
- chocolate (x_3),
- cheese (x_4),
- sour cream (x_5),
- eggs (x_6),
- salmon (x_7),
- pasta (x_8),
- bread (x_9),
- potatoes (x_{10}).

The nutritional values of protein (P), fat (F), carbohydrates (C) [in grams], and food energy [in kcal] per 100 g of each product are taken from a standard cookbook. Average product prices in Poland in 2025 (per 100 g, in euros) are also used, and we assume $const_{BW} = 70$. Based on this data, we can now formulate the MOLP problem:

$$\text{Min } (cost)\ .60x_1 + .67x_2 + .48x_3 + .60x_4 + .17x_5 + .41x_6 + .84x_7 + \\ + .17x_8 + .13x_9 + .09x_{10}$$

$$\text{Max } (energy) 748x_1 + 405x_2 + 560x_3 + 282x_4 + 186x_5 + 140x_6 + \\ + 129x_7 + 368x_8 + 250x_9 + 66x_{10}$$

$$P = 14x_2 + 6x_3 + 26x_4 + 3x_5 + 11x_6 + 13x_7 + 12x_8 + 6x_9 + x_{10}$$
$$F = 82x_1 + 38x_2 + 35x_3 + 18x_4 + 18x_5 + 10x_6 + 9x_7$$
$$C = 60x_3 + 4x_5 + 75x_8 + 55x_9 + 15x_{10}$$
$$2.5P \leq F \leq 3.5P$$
$$0.5P \leq C \leq 0.8P$$
$$P = const_{BW} = 70$$
$$x_1, \ldots, x_{10} \geq 0$$

We analyse the sensitivity of the following efficient solution x^*:

$$x_4^* = 161,\ x_5^* = 811,\ x_8^* = 31,$$
$$x_1^* = x_2^* = x_3^* = x_6^* = x_7^* = x_9^* = x_{10}^* = 0$$

It is worth mentioning that the values of objective functions for this solution are: 2.37 (cost) and 2079 (energy). We present the results of SA using the percentage tolerance approach. Note that in real-life problems the coefficients of the first objective function (cost) may change. Therefore, we allow changes in these coefficients only. To construct matrix H, used in the percentage tolerance approach, we use the following values for the coefficients of H:

$h_{1j} = c_{1j}$ – for $j = 1, \ldots, 10$ – for coefficients of the first objective function (cost)

$h_{2j} = 0$ – for $j = 1, \ldots, 10$ – for the remaining coefficients

In this case, the maximal percentage tolerance for the solutionx^* is $\delta_{max} = 0.094$. This means that x^* remains efficient when cost coefficients vary simultaneously within ±9.4%. To illustrate these results, we calculate the interval for the coefficient $c_{2,3} = 560$ as follows:

$$[c_{2,3} - c_{2,3} \cdot \delta_{max},\ c_{2,3} + c_{2,3} \cdot \delta_{max}] = [560 - 560 \cdot 9.4\%,\ 560 + 560 \cdot$$
$$\cdot 9.4\%] = [507.36,\ 612.64]$$

Analogously, we calculate intervals for the other coefficients.

Remark 5. This remark is analogous to Remark 4. A higher value of $\delta_{\max}$ means a more stable (less sensitive) efficient solution. The interpretation of the obtained intervals is slightly different: they represent possible perturbations of the coefficients in percentage terms. The choice between additive tolerance and percentage tolerance depends on whether the decision maker prefers additive or percentage changes. It is worth noting that the final decision based on additive tolerance may differ from the one based on percentage tolerance.

5 Application in the transportation problem

We consider a multi-criteria transportation problem. This type of problem is studied in numerous papers, including the following: Pandian and Anuradha (2011), Murad et al. (2010). In this paper, we examine a transportation problem based on the paper by Aneya and Nair (1979). The objective is to transport goods from three suppliers (S1, S2, S3) to four warehouses (W1, W2, W3, W4). Two criteria are taken into account:

- minimizing transportation cost,
- minimizing product deterioration.

We present this problem in the form of a bi-criteria linear programming model. All data related to the considered example are presented in Table 3.

Table 3: Data for the transportation problem

(Cost, deteriorations)		Warehouses				Availabilities
		W_1	W_2	W_3	W_4	
Suppliers	S_1	(1, 4)	(2, 4)	(7, 3)	(7, 4)	8
	S_2	(1, 5)	(9, 8)	(3, 9)	(4, 10)	19
	S_3	(8, 6)	(9, 2)	(4, 5)	(6, 1)	17
	Requirements	11	3	14	16	

Let us consider the efficient solution x^*, presented in the form of a transportation plan (Table 4).

Table 4: The considered solution x^* for the transportation problem

5	3	0	0
6	0	0	13
0	0	14	3

We analyze the sensitivity of the efficient solution presented in Table 4. It is worth mentioning that the values of the objective functions for this solution are: 143 (cost) and 265 (deterioration). Moreover, we assume that both cost and deterioration are nonnegative. We apply the method of SA described in Section 2.3, i.e., the standard sensitivity approach. To illustrate the results, we focus on the coefficient $\mathrm{c}_{1,3} = 7$, the cost of transporting goods from the 1st supplier to the 3rd warehouse. By calculating the feasible perturbation of this coefficient, we obtain:

$$T_{x^*} = [0, 10.9]$$

This interval T_{x^*} indicates that the considered solution x^* remains efficient for the following values of the coefficient $c_{1,3}$:

$$0 \leq c_{1,3} \leq 10.9$$

Table 5: Standard SA for individual coefficients – sets T_{x^*}

$\mathbb{R}^+, \mathbb{R}^+$	$\mathbb{R}^+, \mathbb{R}^+$	$[0, 10.9], \mathbb{R}^+$	$[4.2, \infty), \mathbb{R}^+$
$\mathbb{R}^+, \mathbb{R}^+$	$[7.2, 12.3], \mathbb{R}^+$	$\mathbb{R}^+, \mathbb{R}^+$	$\mathbb{R}^+, \mathbb{R}^+$
$[0, 20], \mathbb{R}^+$	$[0, 22.1], \mathbb{R}^+$	$\mathbb{R}^+, \mathbb{R}^+$	$\mathbb{R}^+, \mathbb{R}^+$

Remark 6. The individual changes of the coefficients (T_{x^*}) that do not affect the efficiency of the given solution x^* are described in Table 5. Each interval in Table 5 was obtained under the assumption that all other coefficients remain unchanged. A larger interval indicates a more stable (less sensitive) efficient solution. Note that in Table 5, the second coefficient in each cell always belongs to $\mathbb{R}^+$. This results from the fact that the considered solution x^* is optimal with respect to the objective function representing transportation cost. Moreover, if many efficient solutions are considered, analogous tables (similar to Table 5) can be constructed for each of them. The analysis of such tables is useful for decision-makers for whom sensitivity is an important aspect of the multi-criteria problem.

6 Application in the manufacturing problem (activity-analysis problem)

We consider a well-known linear programming problem: the manufacturing problem. This problem has been analyzed in the single-criterion case, for example, in the papers by Gass (1970; 1975). We extend the single-criterion manufacturing problem (Gass, 1975) to a multi-objective problem in the following way. A manufacturing company has fixed amounts of various resources at its disposal. These resources – such as raw materials, labour, etc. – can be combined to produce any of several different commodities. The company knows how much of each resource is required to produce each commodity. It also knows the profit generated by each product. The company aims to determine a combination of commodities that satisfies multiple objectives:

- maximizing total profit,
- minimizing the usage of selected scarce resources.

The following numerical example is given. A bakery begins the day with a limited supply of flour, shortening, eggs, milk, and yeast. It specializes in producing four types of cakes, and it is assumed that these products can be made in arbitrary amounts (measured in pounds). Additionally, the bakery must produce at least 50 pounds of each type of cake. Two criteria are considered: maximizing profit and minimizing the usage of flour. The recipes are provided in Table 6 (ingredients such as salt, water, and other plentiful supplies are ignored).

Table 6: Manufacturing problem data

	Cake 1	**Cake 2**	**Cake 3**	**Cake 4**	**Available resources**
Shortening	2	12	3	4	4 000
Eggs	0	3	3	1	3 000
Sugar	0.25	1.5	0.125	1	500
Milk	2	0.75	1	0	2 000
Yeast	1	0	1	0	1 000
Flour	12	3	4.5	1.5	MIN
Profit	2	3	4	2	MAX

The MOLP formulation of the considered manufacturing problem is as follows:

$$\begin{gathered}
\text{Max (profit)} 2x_1 + 3x_2 + 4x_3 + 2x_4 \\
\text{Min (flour)} 12x_1 + 3x_2 + 4.5x_3 + 1.5x_4 \\
2x_1 + 12x_2 + 3x_3 + 4x_4 \leq 4\,000 \\
3x_2 + 3x_3 + 1x_4 \leq 3\,000 \\
0.25x_1 + 1.5x_2 + 0.125x_3 + x_4 \leq 500 \\
2x_1 + 0.75x_2 + 1x_3 \leq 2\,000 \\
x_1 + x_3 \leq 1\,000 \\
x_1, x_2 x_3, x_4 \geq 50
\end{gathered}$$

We analyze the sensitivity of the following efficient solution x^*:

$$x_1^* = 50, \;\; x_2^* = 50, \;\; x_3^* = 900, \;\; x_4^* = 150$$

It is worth mentioning that the values of the objective functions for this solution are: 4175 (profit) and 5025 (flour usage). We present the results of SA using the range set approach. Let us note that, as in real-life problems, the profit coefficients in the objective function of our manufacturing model may vary. We assume the existence of a parameter that describes these changes, which are represented using a matrix G (see Section 2.3):

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This means that the first objective function (related to profit) is as follows:

$$\text{(profit)}\; (2+t)x_1 + (3+t)x_2 + (4+t)x_3 + (2+t)x_4$$

Other data in the problem remain unchanged. The range set for the parameter t is $TH(x^*) = [-1, 5]$. This result indicates that the efficient solution x^* remains efficient for all $t \in [-1, 5]$. Moreover, we can calculate the range of values of the objective function related to profit as follows. For $t = -1$ we obtain:

$$\begin{gathered}
(2+t)x_1^* + (3+t)x_2^* + (4+t)x_3^* + (2+t)x_4^* = \\
= (2-1) \cdot 50 + (3-1) \cdot 50 + (4-1) \cdot 900 + (2-1) \cdot 150 = 3000
\end{gathered}$$

Similarly, for $t = 5$, we obtain:

$$(2+5) \cdot 50 + (3+5) \cdot 50 + (4+5) \cdot 900 + (2+5) \cdot 150 = 9900$$

These results illustrate the range of profit values for the given solution $x^*$and the range of $t \in [-1, 5]$. These profit values range from 3000 to 9900.

Remark 7. The range set approach can be also applied in dynamic context. In this case, changes in the parameter t represent changes over time. On the other hand, the range set $TH(x^*)$ illustrates the time interval during which the considered solution x^* remains efficient. Moreover, the obtained range set determines the set of values for the objective functions. If many efficient solutions are considered, a range set can be calculated for each of them. A larger range set indicates a more stable (less sensitive) efficient solution. Therefore, the efficient solution with the largest range set is preferred by a decision-maker for whom changes in the parameter t are significant.

7 Summary

In this paper, we considered multi-objective linear programming (MOLP) problems, where all functions that define the model are linear. We focused on the efficient solutions of MOLP and investigated their sensitivity to changes in the objective function coefficients. The proposed sensitivity analysis (SA) approach aims to determine whether a given efficient solution remains efficient after modifications of certain coefficients of the objective functions. To analyse this, three SA approaches were used:

- The tolerance approach finds one value that shows how much the selected coefficients (possibly all) can be changed.
- The standard approach gives an interval illustrating how much one coefficient of the objective function can change.
- The range set approach, based on parameterizing the objective functions, identifies a set of parameter values that can be applied to the coefficients.

The choice of a suitable approach depends on an analysis of their respective strengths and limitations (see Table 1). To illustrate the proposed methods, four MOLP problems were formulated: the transportation problem, the project management problem, the manufacturing problem, and the diet problem. The results of these applications were discussed and analysed to provide deeper insights into the approaches.

Further research topics are:

- Development of interactive decision-making methods based on the proposed approaches.
- Analysis of the relationships between the presented approaches.
- Formulation of SA approaches using interval parameters, fuzzy numbers, or stochastic parameters.
- Extension of the methods to non-linear multi-objective programming problems.
- Development of SA for the coefficients in the constraints of MOLP problems.
- Application of the approaches to other real-life problems.
- Presentation of the computational aspects of the methods used to compute the values (intervals) in the considered approaches.

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